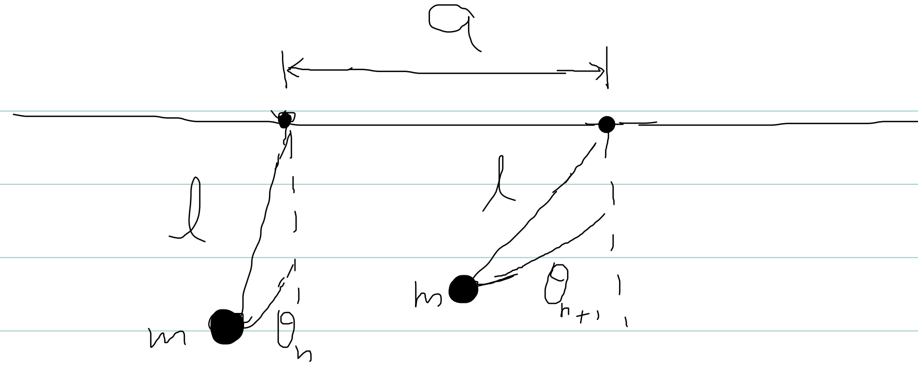
1.



(i) Dynamical variables: there are many equivalent choices. The simplest one is the angular displacement of each mass, θ_n, with respect to the equilibrium position (defined by all penduli masses directly below the wire <u>and</u> the wire being untwisted). We can be even more careful and define the sense of positive displacement, eg, counter-clockwise when looking down the wire in the direction of increasing index "n". We won't need to be that specific. The range of the variables is $\theta_n \in (-\infty, \infty)$. Note that θ_n and $\theta_n + 2\pi$ need not be identified, because holding all other penduli fixed, there is a different twist in the wire between configurations $\{\dots, \theta_{n-1}, \theta_n, \theta_{n+1}, \dots\}$ and $\{\dots, \theta_{n-1}, \theta_n + 2\pi, \theta_{n+1}, \dots\}$.

(ii) Kinetic energy of n-th pendulum: $\frac{1}{2} m (l\dot{\theta}_n)^2$

Gravitational energy " : $mgl(1-\cos\theta_n)$

Elastic energy (twist) in wire between n-th and n+1-st penduli: $\frac{1}{2}\kappa(\theta_{n+1} - \theta_n)^2$

$$\Rightarrow \quad L = \sum_n \left[\frac{1}{2} m l^2 \dot{\theta}_n^2 + mgl\cos\theta_n - \frac{1}{2}\kappa(\theta_{n+1}-\theta_n)^2\right]$$

E.O.M. $\quad \frac{\partial L}{\partial \dot{\theta}_n} = ml^2\dot{\theta}_n \qquad \frac{\partial L}{\partial \theta_n} = -mgl\sin\theta_n - \kappa(\theta_n - \theta_{n-1}) - \kappa(\theta_n - \theta_{n+1})$

$$\Rightarrow \quad -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0 \quad \text{is} \quad ml^2\ddot{\theta}_n + mgl\sin\theta_n - \kappa(\theta_{n+1} + \theta_{n-1} - 2\theta_n) = 0$$

(iii) We want $L \to \int dx\, \mathcal{L}(x)$ when $\theta_n \to \phi(x_n)$ $x_n = na$.

So write $\Delta n = (n+1) - n = 1$ and $\Delta x = a\Delta n$, so

$$\sum_n L_n = \sum_n \Delta n L_n \to \int dx \frac{1}{a} L(na)$$

To this end inspect terms in L/a

$$\frac{1}{a}\left(\frac{m}{2}\ell^2\dot{\theta}_n^2\right) = \frac{1}{2}\frac{m}{a}\ell^2\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2\Big|_{x=na}$$

$$\frac{1}{a}mg\ell\cos\theta_n = \frac{m}{a}g\ell\cos\phi(x,t)\Big|_{x=na}$$

$$-\frac{1}{2}\frac{\kappa}{a}(\theta_{n+1}-\theta_n)^2 = -\frac{1}{2}\kappa a\left(\frac{\phi(x+a,t)-\phi(x,t)}{a}\right)^2\Bigg|_{x=na} \to -\frac{1}{2}\kappa a\left(\frac{\partial\phi}{\partial x}\right)^2\Big|_{x=na}$$

Collecting and taking the limit $a \to 0$ ($N \to \infty$),

$$\mathcal{L} = \frac{1}{2}\left(\frac{m}{a}\right)\ell^2\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{m}{a}g\ell\cos(\phi) - \frac{1}{2}(\kappa a)\left(\frac{\partial\phi}{\partial x}\right)^2$$

Clearly we want $\frac{m\ell^2}{a}$ = fixed, $\frac{mg\ell}{a}$ = fixed and κa = fixed as $a \to 0$.

It is sufficient to take, say, $\frac{m}{a} = \sigma$ and $\kappa a = \rho$, with g and ℓ constant.

EOM: $\mathcal{L} = \mathcal{L}(\partial_t\phi, \partial_x\phi, \phi)$ $\quad \delta S = \int dt\,dx\,\delta\phi\left[-\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} - \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} + \frac{\partial\mathcal{L}}{\partial\phi}\right] = 0$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \left(\frac{m}{a}\right)\ell^2\partial_t\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} = -(\kappa a)\partial_x\phi \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\frac{m}{a}g\ell\sin\phi$$

$$\Longrightarrow -\left(\frac{m}{a}\right)\ell^2\partial_t^2\phi + (\kappa a)\partial_x^2\phi - \frac{m}{a}g\ell\sin\phi = 0 \qquad \text{(EOM)}$$

Hamiltonian density $\mathcal{H} = \pi\partial_t\phi - \mathcal{L}$ with $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \frac{m}{a}\ell^2\partial_t\phi \Rightarrow \partial_t\phi = \frac{a}{m\ell^2}\pi$

$$\mathcal{H} = \frac{1}{2}\frac{a}{m\ell^2}\pi^2 - \frac{m}{a}g\ell\cos\phi + \frac{1}{2}(\kappa a)\left(\frac{\partial\phi}{\partial x}\right)^2$$

(iv) We want to insist that $\mathcal{L}(\phi') = \mathcal{L}(\phi)$ where $\phi'(x) = \phi(\Lambda x)$, where Λ is a Lorentz transformation, $(\Lambda x)^\mu = \begin{pmatrix}\gamma & \gamma\beta \\ \gamma\beta & \gamma\end{pmatrix}\begin{pmatrix}x^0 \\ x^1\end{pmatrix}$ with $\gamma^2 = \frac{1}{1-\beta^2}$ and $x^0 = ct$. That is

$$t' = \gamma\left(t + \beta\frac{x}{c}\right)$$
$$x' = \gamma(\beta ct + x)$$

. We need to compute $\frac{\partial\phi'}{\partial t}$ and $\frac{\partial\phi'}{\partial x}$:

$$\frac{\partial \phi'}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial \phi(t',x')}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial \phi(t',x')}{\partial x} \qquad \frac{\partial \phi'}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial \phi(t',x')}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial \phi(t',x')}{\partial x'}$$

$$= \gamma\, \partial_t \phi + c\gamma\beta\, \partial_x \phi \qquad = \frac{1}{c}\gamma\beta\, \partial_t \phi + \gamma\, \partial_x \phi$$

So
$$\mathcal{L}(\phi) = \frac{1}{2}\frac{m\ell^2}{a}\left(\frac{\partial \phi'}{\partial t}\right)^2 - \frac{1}{2}ka\left(\frac{\partial \phi'}{\partial x}\right)^2 + \frac{m}{a}g\ell\cos\phi'$$

$$= \frac{1}{2}\frac{m\ell^2}{a}\left[\gamma\partial_t\phi + c\gamma\beta\,\partial_x\phi\right]^2 - \frac{1}{2}ka\left[\frac{1}{c}\gamma\beta\,\partial_t\phi + \gamma\,\partial_x\phi\right]^2 + \frac{m}{a}g\ell\cos\phi$$

(all ϕ & $\partial\phi$ evaluated at $x' = \Lambda x$)

$$= \frac{1}{2}\left(\frac{m\ell^2}{a}\gamma^2 - ka\left(\frac{\gamma\beta}{c}\right)^2\right)(\partial_t\phi)^2 - \frac{1}{2}\left(ka\gamma^2 - \frac{m\ell^2}{a}(c\gamma\beta)^2\right)(\partial_x\phi)^2$$

$$+ \left(\frac{m\ell^2}{a}\gamma(c\gamma\beta) - ka\left(\frac{\gamma\beta}{c}\right)\gamma\right)\partial_t\phi\,\partial_x\phi + \frac{m}{a}g\ell\cos\phi$$

This is of the form $A\left(\frac{1}{c^2}(\partial_t\phi)^2 - (\partial_x\phi)^2\right)$ with A a constant, provided

$$\frac{m\ell^2}{a}c\gamma^2\beta - ka\frac{\gamma\beta}{c} = 0 \quad \text{and} \quad \frac{ka\gamma^2 - \frac{m\ell^2}{a}(c\gamma\beta)^2}{\frac{m\ell^2}{a}\gamma^2 - ka\left(\frac{\gamma\beta}{c}\right)^2} = c^2.$$

$\Rightarrow \frac{m\ell^2}{a} = \left(\frac{1}{c^2}\right)ka$. This gives

$$\mathcal{L} = (ka)\left[\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] + \frac{m}{a}g\ell\cos\phi = (ka)\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m}{a}g\ell\cos\phi$$

which is explicitly Lorentz invariant.

2. The point here is that with $\hbar = c = 1$ one has $[\text{Energy}] = [\text{mass}] = 1$
$[\text{length}] = [\text{time}] = -1$. So we can get $[\mathcal{L}]$ and then $[\phi]$ (from $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \cdots$).

(i) Clearly, $[E] = [\text{Lag}] = [\text{Hamiltonian}] = 1$, $L = \int d^3x\, \mathcal{L}$ and $[d^3x] = -3$

$\Rightarrow [\mathcal{L}] = [L] - [d^3x] = 1-(-3) = 4$

Now $[\partial_\mu] = -[x^\mu] = 1$ so $[\partial_\mu\phi\partial_\nu\phi] = 4 \Rightarrow 2[\phi] + 2[\partial] = 4 \Rightarrow [\phi] = 1$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\kappa}{2}[(\partial_\mu\phi)^2]^2 - \frac{m^2}{2}\phi^2 - g\phi^3 - \lambda\phi^4 - \sigma\phi^5$$

so $[\kappa] + 2[(\partial_\mu\phi)^2] = [\mathcal{L}] \Rightarrow [\kappa] = -4$

$[m^2\phi^2] = 4 \Rightarrow [m] = 1$ (no surprise here)

$[g\phi^3] = 4 \Rightarrow [g] = 1$

$[\lambda\phi^4] = 4 \Rightarrow [\lambda] = 0$

$[\sigma\phi^6] = 4 \Rightarrow [\sigma] = -2$

(ii) $[\mathcal{L}] = 1 - [\int d^2x] = 3$, $2[\phi] + 2[\partial_\mu] = 3 \Rightarrow [\phi] = 1/2 \Rightarrow [\kappa] = -3$

$[m] = 1$ (in any number of dimensions, since $[(\partial\phi)^2] = [m^2\phi^2]$ and $[\partial_\mu] = 1$).

$[g] = 3 - 3[\phi] = \frac{3}{2}$

$[\lambda] = 1$

$[\sigma] = 0$

iii) $[\mathcal{L}] = 1 - [\int d^5x] = 6$ $\quad [\phi] = \frac{6-2}{2} = 2$ $\quad [m] = 1$

$[g] = 6 - 3[\phi] = 0$

$[\lambda] = 6 - 4[\phi] = -2$

$[\sigma] = 6 - 6[\phi] = -6$

3. (i) Since $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$

we have $(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma)\eta^{\rho\sigma} = \eta^{\mu\nu}$ to order ω

$\Rightarrow \quad \omega^\mu{}_\rho \delta^\nu{}_\sigma \eta^{\rho\sigma} + \omega^\nu{}_\sigma \delta^\mu{}_\rho \eta^{\rho\sigma} = 0$

Using $\omega^{\mu\sigma} = \omega^\mu{}_\rho \eta^{\rho\sigma}$ this is

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0 \quad \Rightarrow \quad \omega^{\nu\mu} = -\omega^{\mu\nu} \quad \Rightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

(ii) $\epsilon_{\alpha\beta\gamma\delta} X^\alpha{}_\mu X^\beta{}_\nu X^\gamma{}_\rho X^\delta{}_\sigma = -X^0{}_\mu X^1{}_\nu X^2{}_\rho X^3{}_\sigma + X^1{}_\mu X^0{}_\nu X^2{}_\rho X^3{}_\sigma + \dots$

where the terms ... are permutations of 0123 upper indices, with (-1) for even permutations and $(+1)$ for odd permutations. Note that

$\epsilon_{0123} = -1$ since $\epsilon_{0123} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\epsilon^{0123} = (+1)(-1)^3(+1) = -1$.

Now, clearly the above vanishes if any two of the indices $\mu, \nu, \lambda, \sigma$ are equal. So, take $\mu = 0, \nu = 1, \lambda = 2, \sigma = 3$. Then

$\epsilon_{\alpha\beta\gamma\delta} X^\alpha{}_0 X^\beta{}_1 X^\gamma{}_2 X^\delta{}_3 = -X^0{}_0 X^1{}_1 X^2{}_2 X^3{}_3 \pm$ permutations

is precisely $(-1) \times \det(X^\mu{}_\nu)$, by definition of determinant. So

$$\epsilon_{\alpha\beta\gamma\delta} X^\alpha{}_0 X^\beta{}_1 X^\gamma{}_2 X^\delta{}_3 = -\det(X)$$

Since $\mu, \nu, \lambda, \sigma$ have to be all different, and the result is completely antisymmetric under exchange of them, it is proportional to $\epsilon_{\mu\nu\lambda\sigma}$. Hence

$$\epsilon_{\alpha\beta\gamma\delta} X^\alpha{}_\mu X^\beta{}_\nu X^\gamma{}_\rho X^\delta{}_\sigma = \epsilon_{\mu\nu\rho\sigma} \det(X)$$

(iii) Let's do the finite transformations first: the infinitesimal ones will follow.

$$\delta^\mu{}_\nu \to \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma \delta^\rho{}_\sigma = \Lambda^\mu{}_\rho \Lambda_\nu{}^\rho = \delta^\mu{}_\nu$$

The last step follows from the eq. below (1.3) in my course notes:

$\Lambda_\mu{}^\rho$ is the inverse of $\Lambda^\mu{}_\rho$.

Since for finite transformations $\delta^\mu{}_\nu \to \delta^\mu{}_\nu$, the same holds for infinitesimal ones.

Now $\epsilon^{\mu\nu\lambda\sigma} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\lambda{}_\gamma \Lambda^\sigma{}_\delta \; \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\lambda\sigma} \det(\Lambda)$

by the previous item in this problem. So $\epsilon^{\mu\nu\lambda\sigma} \to \pm\, \epsilon^{\mu\nu\lambda\sigma}$ with the sign given by whether $\Lambda \in O^+(1,3)$ or not.

An infinitesimal transformation is in $O^+(1,3)$, so $\epsilon^{\mu\nu\lambda\sigma} \to \epsilon^{\mu\nu\lambda\sigma}$ for infinitesimal transformations.

Note that for more general coordinate transformations, $\Lambda^\mu{}_\nu = \dfrac{\partial x^\mu}{\partial y^\nu}$ where x^μ & y^ν are not necesarily cartesian, so that $\eta^{\mu\nu} \to g^{\mu\nu}$ a metric with the same signature as $\eta^{\mu\nu}$ but with components not necessarily ± 1's, we still have $\delta^\mu{}_\nu \to \delta^\mu{}_\nu$ but not $\epsilon^{\mu\nu\lambda\sigma} \to \epsilon^{\mu\nu\lambda\sigma} \det(\Lambda)$ with $\det(\Lambda)$ nontrivial: $g^{\mu\nu} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma \eta^{\lambda\sigma}$ connects cartesian to general coordinates and $(\det \Lambda)^2 = \det(g)$, $\to$ $\det(\Lambda) = \pm\sqrt{\det(g)}$. So, up to sign, it is $\dfrac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{\det(g)}}$ that is invariant.