Contents

Chapter 3

Stochastic Calculus

3.1 References

– C. Gardiner, *Stochastic Methods* (4^{th} edition, Springer-Verlag, 2010)
Very clear and complete text on stochastic methods, with many applications.

– Z. Schuss, *Theory and Applications of Stochastic Processes* (Springer-Verlag, 2010)
In-depth discussion of continuous path stochastic processes and connections to partial differential equations.

– R. Mahnke, J. Kaupužs, and I. Lubashevsky, *Physics of Stochastic Processes* (Wiley, 2009)
Introductory sections are sometimes overly formal, but a good selection of topics.

– H. Riecke, *Introduction to Stochastic Processes and Stochastic Differential Equations* (unpublished, 2010)
Good set of lecture notes, often following Gardiner. Available online at:
http://people.esam.northwestern.edu/~riecke/Vorlesungen/442/Notes/notes_442.pdf

– J. L. McCauley, *Dynamics of Markets* (2^{nd} edition, Cambridge, 2009)
A physics-friendly discussion of stochastic market dynamics. Crisp and readable. Despite this being the second edition, there are alas a great many typographical errors.

3.2 Gaussian White Noise

Consider a generalized Langevin equation of the form

$$\frac{du}{dt} = f(u,t) + g(u,t)\,\eta(t)\ , \tag{3.1}$$

where $\eta(t)$ is a Gaussian random function with zero mean and

$$\big\langle \eta(t)\,\eta(t')\big\rangle = \phi(t-t')\ . \tag{3.2}$$

The spectral function of the noise is given by the Fourier transform,

$$\hat\phi(\omega) = \int\limits_{-\infty}^{\infty}\!ds\;\phi(s)\,e^{-i\omega s} = \lim_{T\to\infty}\Big\langle \frac{1}{T}\,\big|\hat\eta_T(\omega)\big|^2\Big\rangle\ , \tag{3.3}$$

using the notation of §2.6.3. When $\phi(s) = \varGamma\,\delta(s)$, we have $\hat\phi(\omega) = \varGamma$, *i.e.* independent of frequency. This is the case of *Gaussian white noise*. When $\hat\phi(\omega)$ has a nontrivial dependence on frequency, the noise is said to be *colored*. Gaussian white noise has an infinite variance $\phi(0)$, which leads to problems. In particular, the derivative $\dot u$ strictly speaking does not exist because the function $\eta(t)$ is not continuous.

As an example of the sort of problem this presents, consider the differential equation $\dot u(t) = \eta(t)\,u(t)$. Let's integrate this over a time period Δt from t_j to t_{j+1}, where $t_j = j\,\Delta t$. We then have $u(t_{j+1}) = \big(1+\eta(t_j)\Delta t\big)\,u(t_j)$. Thus, we find

$$u(t_N) = \Big(1+\eta(t_{N-1})\,\Delta t\Big)\cdots\Big(1+\eta(t_0)\,\Delta t\Big)\,u(t_0)\ . \tag{3.4}$$

Now let's compute the average $\big\langle u(t_N)\big\rangle$. Since $\eta(t_j)$ is uncorrelated with $\eta(t_k)$ for all $k\neq j$, we can take the average of each of the terms individually, and since $\eta(t_j)$ has zero mean, we conclude that $\big\langle u(t_N)\big\rangle = u(t_0)$. On average, there is no drift.

Now let's take a continuum limit of the above result, which is to say $\Delta t\to 0$ with $N\Delta t$ finite. Setting $t_0=0$ and $t_N = t$, we have

$$u(t) = u(0)\ \exp\left\{\int\limits_0^t\!ds\;\eta(s)\right\}\ , \tag{3.5}$$

and for Gaussian $\eta(s)$ we have

$$\big\langle u(t)\big\rangle = u(0)\ \exp\left\{\tfrac{1}{2}\int\limits_0^t\!ds\!\int\limits_0^t\!ds'\ \big\langle\eta(s)\,\eta(s')\big\rangle\right\} = u(0)\,e^{\varGamma t/2}\ . \tag{3.6}$$

In the continuum expression, we find there is *noise-induced drift*. The continuum limit of our discrete calculation has failed to match the continuum results. Clearly we have a problem that we must resolve. The origin of the problem is the aforementioned infinite variance of $\eta(t)$. This means that the Langevin equation 3.1 is not well-defined, and in order to get a definite answer we must provide a prescription regarding how it is to be integrated[1].

[1]We will see that Eqn. 3.4 corresponds to the *Itô* prescription and Eqn. 3.5 to the *Stratonovich* prescription.

3.3 Stochastic Integration

3.3.1 Langevin equation in differential form

We can make sense of Eqn. 3.1 by writing it in differential form,

$$du = f(u,t)\,dt + g(u,t)\,dW(t)\ , \tag{3.7}$$

where

$$W(t) = \int\limits_0^t ds\ \eta(s)\ . \tag{3.8}$$

This is because $W(t)$ is described by a Wiener process, for which the sample paths are continuous with probability unity. We shall henceforth take $\Gamma \equiv 1$, in which case $W(t)$ is Gaussianly distributed with $\langle W(t)\rangle = 0$ and

$$\big\langle W(t)\,W(t')\big\rangle = \mathsf{min}(t,t')\ . \tag{3.9}$$

The solution to Eqn. 3.7 is formally

$$u(t) = u(0) + \int\limits_0^t ds\ f\big(u(s),s\big) + \int\limits_0^t dW(s)\ g\big(u(s),s\big)\ . \tag{3.10}$$

Note that Eqn. 3.9 implies

$$\frac{d}{dt'}\,\big\langle W(t)\,W(t')\big\rangle = \Theta(t-t') \quad\Rightarrow\quad \Big\langle \frac{dW(t)}{dt}\,\frac{dW(t')}{dt'}\Big\rangle = \big\langle \eta(t)\,\eta(t')\big\rangle = \delta(t-t') \quad . \tag{3.11}$$

3.3.2 Defining the stochastic integral

Let $F(t)$ be an arbitrary function of time, and let $\{t_j\}$ be a discretization of the interval $[0,t]$ with $j \in \{0,\ldots,N\}$. The simplest example to consider is $t_j = j\,\Delta t$ where $\Delta t = t/N$. Consider the quantity

$$S_N(\alpha) = \sum_{j=0}^{N-1} \Big[(1-\alpha)\,F(t_j) + \alpha\,F(t_{j+1})\Big]\Big[W(t_{j+1}) - W(t_j)\Big]\ , \tag{3.12}$$

where $\alpha \in [0,1]$. Note that the first term in brackets on the RHS can be approximated as

$$F(\tau_j) = (1-\alpha)\,F(t_j) + \alpha\,F(t_{j+1})\ , \tag{3.13}$$

where $\tau_j \equiv (1-\alpha)\,t_j + \alpha\,t_{j+1} \in [t_j, t_{j+1}]$. To abbreviate notation, we will write $F(t_j) = F_j$, $W(t_j) = W_j$, *etc.* We may take $t_0 \equiv 0$ and $W_0 \equiv 0$. The quantities $\Delta W_j \equiv W_{j+1} - W_j$ are *independently and Gaussianly distributed with zero mean for each* j. This means $\big\langle \Delta W_j\big\rangle = 0$ and

$$\big\langle \Delta W_j\,\Delta W_k\big\rangle = \big\langle (\Delta W_j)^2\big\rangle\,\delta_{jk} = \Delta t_j\,\delta_{jk} \quad , \tag{3.14}$$

where $\Delta t_j \equiv t_{j+1} - t_j$. Wick's theorem then tells us

$$\begin{aligned}\big\langle \Delta W_j\,\Delta W_k\,\Delta W_l\,\Delta W_m\big\rangle &= \big\langle \Delta W_j\,\Delta W_k\big\rangle\big\langle \Delta W_l\,\Delta W_m\big\rangle + \big\langle \Delta W_j\,\Delta W_l\big\rangle\big\langle \Delta W_k\,\Delta W_m\big\rangle + \big\langle \Delta W_j\,\Delta W_m\big\rangle\big\langle \Delta W_k\,\Delta W_l\big\rangle \\ &= \Delta t_j\,\Delta t_l\,\delta_{jk}\,\delta_{lm} + \Delta t_j\,\Delta t_k\,\delta_{jl}\,\delta_{km} + \Delta t_j\,\Delta t_k\,\delta_{jm}\,\delta_{kl} \quad .\end{aligned} \tag{3.15}$$

EXERCISE: Show that $\langle W_N^2 \rangle = t$ and $\langle W_N^4 \rangle = 3t^2$.

The expression in Eqn. 3.12 would converge to the integral

$$S = \int_0^t dW(s)\, F(s) \tag{3.16}$$

independent of α were it not for the fact that $\Delta W_j / \Delta t_j$ has infinite variance in the limit $N \to \infty$. Instead, we will find that $S_N(\alpha)$ in general depends on the value of α. For example, the *Itô integral* is defined as the $N \to \infty$ limit of $S_N(\alpha)$ with $\alpha = 0$, whereas the *Stratonovich integral* is defined as the $N \to \infty$ limit of $S_N(\alpha)$ with $\alpha = \frac{1}{2}$.

We now define the stochastic integral

$$\int_0^t dW(s)\, \big[F(s)\big]_\alpha \equiv \text{ms-}\lim_{N\to\infty} \sum_{j=0}^{N-1} \Big[(1-\alpha)\, F(t_j) + \alpha\, F(t_{j+1})\Big]\Big[W(t_{j+1}) - W(t_j)\Big] \quad , \tag{3.17}$$

where ms-lim stands for *mean square limit*. We say that a sequence S_N converges to S in the mean square if $\lim_{N\to\infty} \langle (S_N - S)^2 \rangle = 0$. Consider, for example, the sequence $S_N = \sum_{j=0}^{N-1} (\Delta W_j)^2$. We now take averages, using $\langle (\Delta W_j)^2 \rangle = t_{j+1} - t_j \equiv \Delta t_j$. Clearly $S = \langle S_N \rangle = t$. We also have

$$\langle S_N^2 \rangle = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle (\Delta W_j)^2\, (\Delta W_k)^2 \rangle = (N^2 + 2N)(\Delta t)^2 = t^2 + \frac{2t^2}{N} \quad , \tag{3.18}$$

where we have used Eqn. 3.15. Thus, $\langle (S_N - S)^2 \rangle = 2t^2/N \to 0$ in the $N \to \infty$ limit. So S_N converges to t in the mean square.

Next, consider the case where $F(t) = W(t)$. We find

$$\begin{aligned} S_N(\alpha) &= \sum_{j=0}^{N-1} \Big[(1-\alpha)\, W(t_j) + \alpha\, W(t_{j+1})\Big]\Big[W(t_{j+1}) - W(t_j)\Big] = \sum_{j=0}^{N-1} \big(W_j + \alpha\, \Delta W_j\big)\, \Delta W_j \\ &= \tfrac{1}{2} \sum_{j=0}^{N-1} \Big[\big(W_j + \Delta W_j\big)^2 - W_j^2 + (2\alpha - 1)\big(\Delta W_j\big)^2\Big] = \tfrac{1}{2} W_N^2 + (\alpha - \tfrac{1}{2}) \sum_{j=0}^{N-1} \big(\Delta W_j\big)^2 \ . \end{aligned} \tag{3.19}$$

Taking the average,

$$\langle S_N(\alpha) \rangle = \tfrac{1}{2} t_N + (\alpha - \tfrac{1}{2}) \sum_{j=0}^{N-1} (t_{j+1} - t_j) = \alpha\, t \ . \tag{3.20}$$

Does S_N converge to $\langle S_N \rangle = \alpha t$ in the mean square? Let's define $Q_N \equiv \sum_{j=0}^{N-1} (\Delta W_j)^2$, which is the sequence we analyzed previously. Then $S_N = \frac{1}{2} W_N^2 + (\alpha - \frac{1}{2})\, Q_N$. We then have

$$\langle S_N^2 \rangle = \tfrac{1}{4} \langle W_N^4 \rangle + (\alpha - \tfrac{1}{2})\, \langle W_N^2\, Q_N \rangle + (\alpha - \tfrac{1}{2})^2\, \langle Q_N^2 \rangle \quad , \tag{3.21}$$

with

$$\begin{aligned} \langle W_N^4 \rangle &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \langle \Delta W_j\, \Delta W_k\, \Delta W_l\, \Delta W_m \rangle = 3N^2 (\Delta t)^2 \\ \langle W_N^2\, Q_N \rangle &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \langle \Delta W_j\, \Delta W_k\, (\Delta W_l)^2 \rangle = (N^2 + 2N)(\Delta t)^2 \\ \langle Q_N^2 \rangle &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle (\Delta W_j)^2\, (\Delta W_k)^2 \rangle = (N^2 + 2N)(\Delta t)^2 \quad . \end{aligned} \tag{3.22}$$

Therefore

$$\langle S_N^2 \rangle = \left(\alpha^2 + \tfrac{1}{2}\right) t^2 + \left(\alpha^2 - \tfrac{1}{4}\right) \cdot \frac{2t^2}{N} \quad . \tag{3.23}$$

Therefore $\langle (S_N - \alpha t)^2 \rangle = \frac{1}{2}t^2 + \mathcal{O}(N^{-1})$ and S_N does *not* converge to αt in the mean square! However, if we take

$$S \equiv \int_0^t dW(s)\, \big[W(s)\big]_\alpha = \tfrac{1}{2}W^2(t) + \left(\alpha - \tfrac{1}{2}\right) t \,, \tag{3.24}$$

we have $S_N - S = (\alpha - \frac{1}{2})(Q_N - t)$, S_N converges to S in the mean square. What happened in this example is that $Q_N = \sum_{j=0}^{N-1} (\Delta W_j)^2$ has zero variance in the limit $N \to \infty$, but W_N^2 has finite variance. Therefore S_N has finite variance, and it cannot converge in the mean square to any expression which has zero variance.

3.3.3 Summary of properties of the Itô stochastic integral

For the properties below, it is useful to define the notion of a *nonanticipating function* $F(t)$ as one which is independent of the difference $W(s) - W(t)$ for all $s > t$ at any given t. An example of such a function would be any Itô integral of the form $\int_0^t dW(s)\, G(s)$ or $\int_0^t dW(s)\, G[W(s)]$, where we drop the $[\cdots]_\alpha$ notation since the Itô integral is specified. We then have:[2]

(i) The Itô integral $\int_0^t dW(s)\, F(s)$ exists for all smooth nonanticipating functions $F(s)$.

(ii) $[dW(t)]^2 = dt$ but $[dW(t)]^{2+2p} = 0$ for any $p > 0$. This is because

$$\int_0^t [dW(s)]^2\, F(s) = \text{ms}-\lim_{N\to\infty} \sum_{j=0}^{N-1} F_j\, (\Delta W_j)^2 = \int_0^t ds\, F(s)\,, \tag{3.25}$$

and because $\langle (\Delta W_j)^{2+2p} \rangle \propto (\Delta t)^{1+p}$ for $p > 0$. For the same reason, we may neglect products such as $dt\, dW(t)$.

(iii) We see in (ii) that the m^{th} power of the differential $dW(t)$ is negligible for $m > 2$. If, on the other hand, we take the differential of the m^{th} power of $W(t)$, we obtain

$$\begin{aligned} d[W^m(t)] &= \big[W(t) + dW(t)\big]^m - \big[W(t)\big]^m \\ &= \sum_{k=1}^{m} \binom{m}{k} W^{m-k}(t)\, \big[dW(t)\big]^k \\ &= m\, W^{m-1}(t)\, dW(t) + \tfrac{1}{2}m(m-1)\, W^{m-2}(t)\, dt + o(dt^2)\,. \end{aligned} \tag{3.26}$$

Evaluating the above expression for $m = n+1$ and integrating, we have

$$\begin{aligned} \int_0^t d\big[W^{n+1}(s)\big] &= W^{n+1}(t) - W^{n+1}(0) \\ &= (n+1)\int_0^t dW(s)\, W^n(s) + \tfrac{1}{2}n(n+1)\int_0^t ds\, W^{n-1}(s)\,, \end{aligned} \tag{3.27}$$

[2] See Gardiner §4.2.7.

and therefore

$$\int_0^t dW(s)\, W^n(s) = \frac{W^{n+1}(t) - W^{n+1}(0)}{n+1} - \tfrac{1}{2} n \int_0^t ds\, W^{n-1}(s) \ . \tag{3.28}$$

(iv) Consider the differential of a function $f\big[W(t),t\big]$:

$$\begin{aligned} df\big[W(t),t\big] &= \frac{\partial f}{\partial W}\, dW + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\,\frac{\partial^2 f}{\partial W^2}\,(dW)^2 + \frac{\partial^2 f}{\partial W\,\partial t}\, dW\, dt + \frac{1}{2}\,\frac{\partial^2 f}{\partial t^2}\,(dt)^2 + \ldots \\ &= \bigg(\frac{\partial f}{\partial t} + \frac{1}{2}\,\frac{\partial^2 f}{\partial W^2}\bigg) dt + \frac{\partial f}{\partial W}\, dW + o(dt) \ . \end{aligned} \tag{3.29}$$

For example, for $f = \exp(W)$, we have $d\,e^{W(t)} = e^{W(t)}\big(dW(t) + \frac{1}{2}dt\big)$. This is known as *Itô's formula*. As an example of the usefulness of Itô's formula, consider the function $f\big[W(t),t\big] = W^2(t) - t$, for which Itô's formula yields $df = 2W\,dW$. Integrating the differential df, we thereby recover the result,

$$\int_0^t dW(s)\, W(s) = \tfrac{1}{2} W^2(t) - \tfrac{1}{2} t \ . \tag{3.30}$$

(v) If $F(t)$ is nonanticipating, then

$$\bigg\langle \int_0^t dW(s)\, F(s) \bigg\rangle = 0 \ . \tag{3.31}$$

Again, this is true for the Itô integral but not the Stratonovich integral.

(vi) The correlator of two Itô integrals of nonanticipating functions $F(s)$ and $G(s')$ is given by

$$\bigg\langle \int_0^t dW(s)\, F(s) \int_0^{t'} dW(s')\, G(s') \bigg\rangle = \int_0^{\tilde t} ds\, F(s)\, G(s) \ , \tag{3.32}$$

where $\tilde t = \mathsf{min}(t,t')$. This result was previously obtained by writing $dW(s) = \eta(s)\, ds$ and then invoking the correlator $\big\langle \eta(s)\,\eta(s')\big\rangle = \delta(s-s')$.

(vii) Oftentimes we encounter stochastic integrals in which the integrand contains a factor of $\delta(t-t_1)$ or $\delta(t-t_2)$, where the range of integration is the interval $[t_1,t_2]$. Appealing to the discretization defined in §3.3.2, it is straightforward to show

$$\begin{aligned} I_1 &= \int_{t_1}^{t_2} dt\, f(t)\, \delta(t-t_1) = (1-\alpha)\, f(t_1) \\ I_2 &= \int_{t_1}^{t_2} dt\, f(t)\, \delta(t-t_2) = \alpha\, f(t_2) \ . \end{aligned} \tag{3.33}$$

Thus, for Itô, $I_1 = f(t_1)$ and $I_2 = 0$, whereas for Stratonovich $I_1 = \frac{1}{2}\, f(t_1)$ and $I_2 = \frac{1}{2}\, f(t_2)$.

3.3.4 Fokker-Planck equation

We saw in §2.4 how the drift and diffusion relations

$$\big\langle \delta u(t) \big\rangle = F_1\big(u(t)\big)\,\delta t \qquad , \qquad \big\langle \big[\delta u(t)\big]^2 \big\rangle = F_2\big(u(t)\big)\,\delta t \,, \tag{3.34}$$

where $\delta u(t) = u(t+\delta t) - u(t)$, results in a Fokker-Planck equation

$$\frac{\partial P(u,t)}{\partial t} = -\frac{\partial}{\partial u}\Big[F_1(u)\,P(u,t)\Big] + \frac{1}{2}\,\frac{\partial^2}{\partial u^2}\Big[F_2(u)\,P(u,t)\Big] \,. \tag{3.35}$$

Consider now the differential Langevin equation

$$du = f(u,t)\,dt + g(u,t)\,dW(t) \,. \tag{3.36}$$

Let's integrate over the interval $[0,t]$, and work only to order t in $u(t) - u_0$, where $u_0 \equiv u(0)$. We then have

$$\begin{aligned} u(t) - u_0 &= \int\limits_0^t \! ds\; f\big(u(s)\big) + \int\limits_0^t \! dW(s)\; g\big(u(s)\big) \\ &= f(u_0)\,t + g(u_0) \int\limits_0^t \! dW(s) + g'(u_0) \int\limits_0^t \! dW(s)\,\big[u(s) - u_0\big] + \ldots \\ &= f(u_0)\,t + g(u_0)\,W(t) + f(u_0)\,g'(u_0) \int\limits_0^t \! dW(s)\; s + g(u_0)\,g'(u_0) \int\limits_0^t \! dW(s)\,W(s) + \ldots \,, \end{aligned} \tag{3.37}$$

where $W(t) = \int\limits_0^t ds\,\eta(s) = 0$, hence $W(0) = 0$. Averaging, we find

$$\big\langle u(t) - u_0 \big\rangle = f(u_0)\,t + \alpha\,g(u_0)\,g'(u_0)\,t + \ldots \tag{3.38}$$

and

$$\big\langle \big[u(t) - u_0\big]^2 \big\rangle = g^2(u_0)\,t + \ldots \tag{3.39}$$

After a brief calculation, we obtain

$$\begin{aligned} F_1(u) &= f(u) + \alpha\,g(u)\,g'(u) \\ F_2(u) &= g^2(u) \,. \end{aligned} \tag{3.40}$$

We see how, for any choice other than the Itô value $\alpha = 0$, there is a noise-induced drift.

3.4 Stochastic Differential Equations

The general form we are considering is

$$du = \; f(u,t)\,dt + g(u,t)\,dW \,. \tag{3.41}$$

This is a *stochastic differential equation* (SDE). We are here concerned with (i) change of variables, (ii) multivariable formulations, and (iii) differences between Itô and Stratonovich solutions.

3.4.1 Itô change of variables formula

Suppose we change variables from u to $v(u,t)$. Then

$$\begin{aligned} dv &= \frac{\partial v}{\partial t}\,dt + \frac{\partial v}{\partial u}\,du + \frac{1}{2}\,\frac{\partial^2 v}{\partial u^2}\,(du)^2 + o(dt) \\ &= \bigg(\frac{\partial v}{\partial t} + f\,\frac{\partial v}{\partial u} + \tfrac{1}{2}g^2\,\frac{\partial^2 v}{\partial u^2}\bigg)dt + g\,\frac{\partial v}{\partial u}\,dW + o(dt)\ , \end{aligned} \tag{3.42}$$

where we have used $(dW)^2 = dt$. Note that if $v = v(u)$ we do not have the $\frac{\partial v}{\partial t}\,dt$ term. This change of variables formula is only valid for the Itô case. In §3.4.5 below, we will derive the corresponding result for the Stratonovich case, and show that it satisfies the familiar chain rule.

EXERCISE: Derive the change of variables formula for general α. *Hint:* First integrate the SDE over a small but finite time interval Δt_j to obtain

$$\begin{aligned} \Delta u_j &= f_j\,\Delta t_j + \big[(1-\alpha)\,g_j + \alpha\,g_{j+1}\big]\,\Delta W_j \\ &= \Big[f_j + \alpha\,g_j\,g_j'\Big]\Delta t_j + g_j\,\Delta W_j \quad , \end{aligned} \tag{3.43}$$

up to unimportant terms, where $u_j = u(t_j)$, $f_j = f(u_j,t_j)$, $g_j = g(u_j,t_j)$, and $g_j' = \frac{\partial g}{\partial u}\big|_{u_j,t_j}$.

Example: Kubo oscillator

As an example, consider the Kubo oscillator[3],

$$du = i\omega\,u\,dt + i\lambda\,u\,dW\ . \tag{3.44}$$

This can be interpreted as a linear oscillator with a fluctuating frequency. If $\lambda = 0$, we have $\dot{u} = i\omega u$, with solution $u(t) = u(0)\,e^{i\omega t}$. We now implement two changes of variables:

(i) First, we define $v = u\,e^{-i\omega t}$. Plugging this into Eqn. 3.42, we obtain

$$dv = i\lambda\,v\,dW\ . \tag{3.45}$$

(ii) Second, we write $y = \ln v$. Appealing once again to the Itô change of variables formula, we find

$$dy = \tfrac{1}{2}\lambda^2\,dt + i\lambda\,dW\ . \tag{3.46}$$

The solution is therefore

$$y(t) = y(0) + \tfrac{1}{2}\lambda^2 t + i\lambda\,W(t) \qquad \Longrightarrow \qquad u(t) = u(0)\,e^{i\omega t}\,e^{\lambda^2 t/2}\,e^{i\lambda W(t)}\ . \tag{3.47}$$

Averaging over the Gaussian random variable W, we have

$$\big\langle u(t)\big\rangle = u(0)\,e^{i\omega t}\,e^{\lambda^2 t/2}\,e^{-\lambda^2\langle W^2(t)\rangle/2} = u(0)\,e^{i\omega t}\ . \tag{3.48}$$

Thus, the average of $u(t)$ behaves as if it is unperturbed by the fluctuating piece. *There is no noise-induced drift.* We can also compute the correlator,

$$\big\langle u(t)\,u^*(t')\big\rangle = \big|u(0)\big|^2\,e^{i\omega(t-t')}\,e^{\lambda^2 \mathsf{min}(t,t')}\ . \tag{3.49}$$

Thus, $\big\langle \big|u(t)\big|^2\big\rangle = \big|u(0)\big|^2\,e^{\lambda^2 t}$. If $u(0)$ is also a stochastic variable, we must average over it as well.

[3]See Riecke, §5.4.1 and Gardiner §4.5.3.

3.4.2 Solvability by change of variables

Following Riecke[4], we ask under what conditions the SDE $du = f(u,t)\,dt + g(u,t)\,dW$ can be transformed to

$$dv = \alpha(t)\,dt + \beta(t)\,dW\ , \tag{3.50}$$

which can be directly integrated via Itô. From Itô's change of variables formula Eqn. 3.42, we have

$$dv = \bigg(\frac{\partial v}{\partial t} + f\,\frac{\partial v}{\partial u} + \tfrac{1}{2}g^2\,\frac{\partial^2 v}{\partial u^2}\bigg)dt + g\,\frac{\partial v}{\partial u}\,dW\ , \tag{3.51}$$

hence

$$\alpha(t) = \frac{\partial v}{\partial t} + f\,\frac{\partial v}{\partial u} + \tfrac{1}{2}g^2\,\frac{\partial^2 v}{\partial u^2} \qquad , \qquad \beta(t) = g\,\frac{\partial v}{\partial u} \quad . \tag{3.52}$$

We therefore have

$$\frac{\partial v}{\partial u} = \frac{\beta(t)}{g(u,t)} \qquad \Rightarrow \qquad \frac{\partial^2 v}{\partial u^2} = -\frac{\beta}{g^2}\,\frac{\partial g}{\partial u} \qquad , \qquad \frac{\partial^2 v}{\partial u\,\partial t} = \frac{1}{g}\,\frac{d\beta}{dt} - \frac{\beta}{g^2}\,\frac{\partial g}{\partial t}\ . \tag{3.53}$$

Setting $\partial\alpha/\partial u = 0$ then results in

$$\frac{1}{g}\,\frac{d\beta}{dt} - \frac{\beta}{g^2}\,\frac{\partial g}{\partial t} + \frac{\partial}{\partial u}\bigg[\beta\,\frac{f}{g} - \tfrac{1}{2}\beta\,\frac{\partial g}{\partial u}\bigg] = 0\ , \tag{3.54}$$

or

$$\frac{d\ln\beta}{dt} = \frac{\partial \ln g}{\partial t} - g\,\frac{\partial}{\partial u}\bigg(\frac{f}{g}\bigg) + \tfrac{1}{2}\,g\,\frac{\partial^2 g}{\partial u^2}\ . \tag{3.55}$$

The LHS of the above equation is a function of t alone, hence the solvability condition becomes

$$\frac{\partial}{\partial u}\Bigg[\frac{\partial \ln g}{\partial t} - g\,\frac{\partial}{\partial u}\bigg(\frac{f}{g}\bigg) + \tfrac{1}{2}\,g\,\frac{\partial^2 g}{\partial u^2}\Bigg] = 0\ . \tag{3.56}$$

If the above condition holds, one can find a u-independent $\beta(t)$, and from the second of Eqn. 3.52 one then obtains $\partial v/\partial u$. Plugging this into the first of Eqn. 3.52 then yields $\alpha(t)$, which is itself guaranteed to be u-independent.

3.4.3 Multicomponent SDE

Let $\boldsymbol{u} = \{u_1, \ldots, u_K\}$ and consider the SDE

$$du_a = A_a\,dt + B_{ab}\,dW_b\ , \tag{3.57}$$

where repeated indices are summed over, and where

$$\big\langle dW_b\,dW_c\big\rangle = \delta_{bc}\,dt\ . \tag{3.58}$$

Now suppose $f(\boldsymbol{u})$ is a scalar function of the collection $\{u_1, \ldots, u_K\}$. We then have

$$\begin{aligned} df &= \frac{\partial f}{\partial u_a}\,du_a + \frac{1}{2}\,\frac{\partial^2 f}{\partial u_a\,\partial u_b}\,du_a\,du_b + o(dt) \\ &= \frac{\partial f}{\partial u_a}\big(A_a\,dt + B_{ab}\,dW_b\big) + \frac{1}{2}\,\frac{\partial^2 f}{\partial u_a\,\partial u_b}\big(A_a\,dt + B_{aa'}\,dW_{a'}\big)\big(A_b\,dt + B_{bb'}\,dW_{b'}\big) + o(dt) \\ &= \bigg[A_a\,\frac{\partial f}{\partial u_a} + \frac{1}{2}\,\frac{\partial^2 f}{\partial u_a\,\partial u_b}\,\big(BB^{\rm t}\big)_{ba}\bigg]dt + \frac{\partial f}{\partial u_a}\,B_{ab}\,dW_b + o(dt)\ . \end{aligned} \tag{3.59}$$

[4] See Riecke, §5.4.2.

We also may derive the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u_a}\big(A_a P\big) + \frac{1}{2}\,\frac{\partial^2}{\partial u_a\,\partial u_b}\Big[(BB^{\rm t})_{ab} P\Big] \ . \tag{3.60}$$

3.4.4 SDEs with general α expressed as Itô SDEs ($\alpha = 0$)

We return to the single component case and the SDE

$$du = f(u,t)\,dt + g(u,t)\,dW(t) \quad . \tag{3.61}$$

Formally, we can write

$$u(t) - u(0) = \int\limits_0^t \!ds\; f\big(u(s),s\big) + \int\limits_0^t \!dW(s)\; g\big(u(s),s\big) \ . \tag{3.62}$$

The second term on the RHS is defined via its discretization, with

$$\begin{aligned}\int\limits_0^t \!dW(s)\,\Big[g\big(u(s),s\big)\Big]_\alpha &\equiv \text{ms}-\lim_{N\to\infty}\sum_{j=0}^{N-1} g\big((1-\alpha)u_j + \alpha\,u_{j+1}, t_j\big)\,\Delta W_j \\ &= \text{ms}-\lim_{N\to\infty}\sum_{j=0}^{N-1}\Big[g(u_j,t_j)\,\Delta W_j + \alpha\,\frac{\partial g}{\partial u}(u_j,t_j)\,(u_{j+1}-u_j)\,\Delta W_j\Big] \ .\end{aligned} \tag{3.63}$$

Now if u satisfies the SDE $du = f\,dt + g\,dW$, then

$$u_{j+1} - u_j = f(u_j,t_j)\,\Delta t_j + g(u_j,t_j)\,\Delta W_j \ , \tag{3.64}$$

where $\Delta t_j = t_{j+1} - t_j$, and inserting this into the previous equation gives

$$\begin{aligned}\int\limits_0^t \!dW(s)\,\Big[g\big(u(s),s\big)\Big]_\alpha &= \text{ms}-\lim_{N\to\infty}\sum_{j=0}^{N-1}\Big[g(u_j,t_j)\,\Delta W_j + \alpha\,f(u_j,t_j)\,\frac{\partial g}{\partial u}(u_j,t_j)\,\Delta t_j\,\Delta W_j + \alpha\,g(u_j,t_j)\,\frac{\partial g}{\partial u}(u_j,t_j)\,(\Delta W_j)^2\Big] \\ &= \int\limits_0^t \!dW(s)\,\Big[g\big(u(s),s\big)\Big]_0 + \alpha\int\limits_0^t \!ds\; g\big(u(s),s\big)\,\frac{\partial g}{\partial u}\big(u(s),s\big) \ ,\end{aligned} \tag{3.65}$$

where the stochastic integral with $\alpha = 0$ found on the last line above is the Itô integral. Thus, the solution of the stochastic differential equation Eqn. 3.61, using the prescription of stochastic integration for general α, is equivalent to the solution using the Itô prescription ($\alpha = 0$) if we substitute

$$f_{\rm I}(u,t) = f(u,t) + \alpha\,g(u,t)\,\frac{\partial g(u,t)}{\partial u} \qquad , \qquad g_{\rm I}(u,t) = g(u,t) \ , \tag{3.66}$$

where the I subscript denotes the Itô case. In particular, since $\alpha = \frac{1}{2}$ for the Stratonovich case,

$$\begin{aligned} du = f\,dt + g\,dW \quad \text{[Itô]} \qquad &\Longrightarrow \qquad du = \Big(f - \tfrac{1}{2}\,g\,\frac{\partial g}{\partial u}\Big)\,dt + g\,dW \quad \text{[Stratonovich]} \\ du = f\,dt + g\,dW \quad \text{[Stratonovich]} \qquad &\Longrightarrow \qquad du = \Big(f + \tfrac{1}{2}\,g\,\frac{\partial g}{\partial u}\Big)\,dt + g\,dW \quad \text{[Itô]} \ .\end{aligned}$$

Kubo oscillator as a Stratonovich SDE

Consider the case of the Kubo oscillator, for which $f = i\omega u$ and $g = i\lambda u$. Viewed as a Stratonovich SDE, we transform to Itô form to obtain

$$du = \big(i\omega - \tfrac{1}{2}\lambda^2\big)u\,dt + i\lambda u\,dW \ . \tag{3.67}$$

Solving as in §3.4.1, we find

$$u(t) = u(0)\,e^{i\omega t}\,e^{i\lambda W(t)} \ , \tag{3.68}$$

hence

$$\big\langle u(t)\big\rangle = u(0)\,e^{i\omega t}\,e^{-\lambda^2 t/2} \qquad , \qquad \big\langle u(t)\,u^*(t')\big\rangle = \big|u(0)\big|^2\,e^{i\omega(t-t')}\,e^{-\lambda^2|t-t'|/2} \ . \tag{3.69}$$

We see that there is noise-induced drift and decay in the Stratonovich case.

Multivariable case

Suppose we have

$$\begin{aligned} du_a &= A_a\,dt + B_{ab}\,dW_b \quad (\alpha\text{−discretization}) \\ &= \widetilde{A}_a\,dt + \widetilde{B}_{ab}\,dW_b \quad (\text{Itô}) \ . \end{aligned} \tag{3.70}$$

Using $\langle\, dW_a\, dW_b\,\rangle = \delta_{ab}\,dt$, applying the above derivation in §3.4.3, we obtain

$$\widetilde{A}_a = A_a + \alpha\,\frac{\partial B_{ac}}{\partial u_b}\,B^{\rm t}_{cb} \qquad , \qquad \widetilde{B}_{ab} = B_{ab} \ , \tag{3.71}$$

where repeated indices are summed. The resulting Fokker-Planck equation is then

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u_a}\bigg[\bigg(A_a + \alpha\,\frac{\partial B_{ac}}{\partial u_b}\,B^{\rm t}_{cb}\bigg)P\bigg] + \frac{1}{2}\,\frac{\partial^2}{\partial u_a\,\partial u_b}\Big[\big(BB^{\rm t}\big)_{ab}\,P\Big] \ . \tag{3.72}$$

When $\alpha = \frac{1}{2}$, we obtain the Stratonovich form,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u_a}\Big(A_a\,P\Big) + \frac{1}{2}\,\frac{\partial}{\partial u_a}\bigg[B_{ac}\,\frac{\partial}{\partial u_b}\Big(B^{\rm t}_{cb}\,P\Big)\bigg] \ . \tag{3.73}$$

3.4.5 Change of variables in the Stratonovich case

We saw in Eqn. 3.42 how a change of variables leads to a new SDE in the Itô case. What happens in the Stratonovich case? To see this, we write the Stratonovich SDE,

$$du = f\,dt + g\,dW \ , \tag{3.74}$$

in its Itô form,

$$du = \bigg(f + \tfrac{1}{2}\,g\,\frac{\partial g}{\partial u}\bigg)\,dt + g\,dW \ , \tag{3.75}$$

and now effect the change of variables $v = v(u)$. We leave the general case of $v = v(u,t)$ to the student. Applying Eqn. 3.42, we find

$$\begin{aligned} dv &= \bigg[\bigg(f + \frac{1}{2}\,g\,\frac{\partial g}{\partial u}\bigg)\,\frac{dv}{du} + \frac{1}{2}\,\frac{d^2 v}{du^2}\,g^2\bigg]dt + \frac{\partial v}{\partial u}\,g\,dW \\ &= \bigg[\frac{f}{u'} + \frac{1}{2}\,\frac{\partial g}{\partial v}\,\frac{g}{(u')^2} - \frac{1}{2}\,\frac{g^2\,u''}{(u')^3}\bigg]dt + \frac{g}{u'}\,dW \ , \end{aligned} \tag{3.76}$$

where $u' = du/dv$ and $u'' = d^2u/dv^2$. Now that everything in the last line above is expressed in terms of v and t, we transform back to the Stratonovich form, resulting in

$$dv = \tilde{f}\, dt + \tilde{g}\, dW \ , \tag{3.77}$$

with

$$\tilde{f} = \frac{f}{u'} + \frac{1}{2}\frac{\partial g}{\partial v}\frac{g}{(u')^2} - \frac{1}{2}\frac{g^2\, u''}{(u')^3} - \frac{1}{2}\bigg(\frac{g}{u'}\bigg)\frac{\partial}{\partial v}\bigg(\frac{g}{u'}\bigg) = \frac{f}{u'} \tag{3.78}$$

and

$$\tilde{g} = \frac{g}{u'} \ . \tag{3.79}$$

Thus,

$$dv = \frac{1}{u'}\Big[f\, dt + g\, dW\Big] = \frac{dv}{du}\, du \ , \tag{3.80}$$

which satisfies the familiar chain rule!

3.5 Applications

3.5.1 Ornstein-Uhlenbeck redux

The Ornstein-Uhlenbeck process is described by the SDE

$$dx = -\beta\, x\, dt + \sqrt{2D}\, dW(t) \ . \tag{3.81}$$

Since the coefficient of dW is independent of x, this equation is the same when the Itô prescription is taken. Changing variables to $y = x\, e^{\beta t}$, we have

$$dy = \sqrt{2D}\, e^{\beta t}\, dW(t) \ , \tag{3.82}$$

with solution

$$x(t) = x(0)\, e^{-\beta t} + \sqrt{2D}\int\limits_0^t dW(s)\, e^{-\beta(t-s)} \ . \tag{3.83}$$

We may now compute

$$\big\langle x(t)\big\rangle = x(0)\, e^{-\beta t} \quad , \quad \Big\langle\big(x(t) - x(0)\, e^{-\beta t}\big)^2\Big\rangle = \frac{D}{\beta}\left(1 - e^{-2\beta t}\right) . \tag{3.84}$$

The correlation function is also easily calculable:

$$\begin{aligned}\big\langle x(t)\, x(t')\big\rangle_{\rm c} &= \big\langle x(t)\, x(t')\big\rangle - \big\langle x(t)\big\rangle\big\langle x(t')\big\rangle \\ &= 2D\,\Bigg\langle \int\limits_0^t dW(s)\, e^{-\beta(t-s)} \int\limits_0^{t'} dW(s')\, e^{-\beta(t'-s')}\Bigg\rangle \\ &= 2D\, e^{-\beta(t+t')}\int\limits_0^{\min(t,t')} ds\, e^{2\beta s} = \frac{D}{\beta}\Big(e^{-\beta|t-t'|} - e^{-\beta(t+t')}\Big) \ .\end{aligned} \tag{3.85}$$

3.5.2 Time-dependence

Consider the SDE,

$$du = \alpha(t)\, u\, dt + \beta(t)\, u\, dW(t) \ . \tag{3.86}$$

Writing $v = \ln u$ and appealing the the Itô change of variables formula in Eqn. 3.42, we have

$$dv = \Big(\alpha(t) - \tfrac{1}{2}\beta^2(t)\Big) dt + \beta(t)\, dW(t) \ , \tag{3.87}$$

which may be directly integrated to yield

$$u(t) = u(0)\ \exp\left\{ \int\limits_0^t \! ds \left[\alpha(s) - \tfrac{1}{2}\beta^2(s)\right] + \int\limits_0^t \! dW(s)\ \beta(s) \right\} \ . \tag{3.88}$$

Using the general result for the average of the exponential of a Gaussian random variable, $\big\langle \exp(\phi)\big\rangle = \exp\big(\frac{1}{2}\langle \phi^2\rangle\big)$, we have

$$\big\langle u^n(t)\big\rangle = u^n(0)\ \exp\left\{ \int\limits_0^t \! ds \left[n\,\alpha(s) + \tfrac{1}{2} n(n-1)\,\beta^2(s)\right] \right\} \ . \tag{3.89}$$

3.5.3 Colored noise

We can model colored noise using the following artifice[5]. We saw above how the Ornstein-Uhlenbeck process yields a correlation function

$$C(s) = \big\langle u(t)\, u(t+s)\big\rangle = \frac{D}{\beta}\, e^{-\beta |s|} \ , \tag{3.90}$$

in the limit $t\to\infty$. This means that the spectral function is

$$\hat{C}(\omega) = \int\limits_{-\infty}^{\infty} \! ds\ C(s)\, e^{-i\omega s} = \frac{2D}{\beta^2 + \omega^2} \ , \tag{3.91}$$

which has spectral variation. We henceforth set $2D \equiv \beta^2$ so that $C(s) = \frac{1}{2}\beta\, e^{-\beta|s|}$, and $\hat{C}(\omega) = \beta^2/(\beta^2+\omega^2)$. Note that $\hat{C}(0) = \int\limits_{-\infty}^{\infty} ds\ C(s) = 1$.

Consider now a quantity $x(t)$ which is *driven* by the OU process, *viz.*

$$\begin{aligned} du &= -\beta\, u\, dt + \beta\, dW(t) \\ \frac{dx}{dt} &= a(t)\, x + b(t)\, u(t)\, x \ , \end{aligned} \tag{3.92}$$

where $a(t)$ and $b(t)$ may be time-dependent. The second of these is an ordinary differential equation and not a SDE since $u(t)$ is a continuous function, even though it is stochastic. As we saw above, the solution for $u(t)$ is

$$u(t) = u(0)\, e^{-\beta t} + \beta \int\limits_0^t \! dW(s)\ e^{-\beta(t-s)} \ . \tag{3.93}$$

[5]See Riecke §5.6.

Therefore

$$x(t) = x(0)\,\exp\left\{\int\limits_0^t ds\,a(s) + u(0)\int\limits_0^t ds\,b(s)\,e^{-\beta s} + \beta\int\limits_0^t ds\,b(s)\int\limits_0^s dW(s')\,e^{-\beta(s-s')}\right\}\ . \tag{3.94}$$

It is convenient to reexpress the last term in brackets such that

$$x(t) = x(0)\,\exp\left\{\int\limits_0^t ds\,a(s) + u(0)\int\limits_0^t ds\,b(s)\,e^{-\beta s} + \beta\int\limits_0^t dW(s')\int\limits_{s'}^t ds\,b(s)\,e^{-\beta(s-s')}\right\}\ . \tag{3.95}$$

Now let us take the $\beta \to \infty$ limit. We know that for any smooth function $f(s)$ that

$$\lim_{\beta\to\infty}\beta\int\limits_{s'}^t ds\,b(s)\,e^{-\beta(s-s')} = b(s')\ , \tag{3.96}$$

hence

$$\lim_{\beta\to\infty} x(t) = x(0)\,\exp\left\{\int\limits_0^t ds\,a(s) + \int\limits_0^t dW(s)\,b(s)\right\}\ . \tag{3.97}$$

Now since $\big\langle u(t)\,u(t')\big\rangle = C(t-t') = \delta(t-t')$ in the $\beta \to \infty$ limit, we might as well regard $x(t)$ as being stochastically forced by a Wiener process and describe its evolution using the SDE,

$$dx = a(t)\,x\,dt + b(t)\,x\,dW(t) \qquad (\alpha = ??)\ . \tag{3.98}$$

As we have learned, the integration of SDEs is a negotiable transaction, which requires fixing a value of the interval parameter α. What value of α do we mean for the above equation? We can establish this by transforming it to an Itô SDE with $\alpha = 0$, using the prescription in Eqn. 3.66. Thus, with α as yet undetermined, the Itô form of the above equation is

$$dx = \Big[a(t) + \alpha\,b^2(t)\Big]\,x\,dt + b(t)\,x\,dW(t)\ . \tag{3.99}$$

Now we use the Itô change of variables formula 3.42 to write this as a SDE for $y = \ln x$:

$$dy = \Big[a(t) + (\alpha - \tfrac{1}{2})\,b^2(t)\Big]\,dt + b(t)\,dW(t)\ , \tag{3.100}$$

which may be integrated directly, yielding

$$x(t) = x(0)\,\exp\left\{\int\limits_0^t ds\,\Big[a(s) + (\alpha - \tfrac{1}{2})\,b^2(s)\Big] + \int\limits_0^t dW(s)\,b(s)\right\}\ . \tag{3.101}$$

Comparing with Eqn. 3.97, we see that $\alpha = \frac{1}{2}$, *i.e.* Stratonovich form.

Finally, what of the correlations? Consider the case where $a(t) \to i\nu$ and $b(t) \to i\lambda$ are complex constants, in which case we have a colored noise version of the Kubo oscillator:

$$\begin{aligned} du &= -\beta\,u\,dt + \beta\,dW(t) \\ \frac{dz}{dt} &= i\nu\,z + i\lambda\,u(t)\,z\ . \end{aligned} \tag{3.102}$$

The solution is

$$z(t) = z(0)\,\exp\left\{i\nu t + \frac{i\lambda}{\beta}\,u(0)\,\big(1 - e^{-\beta t}\big) + i\lambda\int\limits_0^t dW(s)\,\Big(1 - e^{-\beta(t-s)}\Big)\right\}\ . \tag{3.103}$$

This matches the Stratonovich solution to the Kubo oscillator, $z(t) = z(0)\, e^{i\nu t}\, e^{i\lambda W(t)}$ in the limit $\beta \to \infty$, as we should by now expect. The average oscillator coordinate is

$$\big\langle z(t)\big\rangle = z(0)\, \exp\left\{ i\nu t + \frac{i\lambda}{\beta}\, u(0)\,\big(1 - e^{-\beta t}\big) - \tfrac{1}{2}\lambda^2 t + \frac{\lambda^2}{2\beta}\big(1 - e^{-\beta t}\big)^2 \right\} . \tag{3.104}$$

As $\beta \to \infty$ we recover the result from Eqn. 3.69. For $\beta \to 0$, the stochastic variable $u(t)$ is fixed at $u(0)$, and $z(t) = z(0)\, \exp\big(i[\nu + \lambda\, u(0)]\, t\big)$, which is correct.

Let's now compute the correlation function $\big\langle z(t)\, z^*(t')\big\rangle$ in the limit $t, t' \to \infty$, where it becomes a function of $t - t'$ alone due to decay of the transients arising from the initial conditions. It is left as an exercise to the reader to show that

$$Y(s) = \lim_{t\to\infty} \big\langle z(t+s)\, z^*(t)\big\rangle = |z(0)|^2\, \exp\left\{ i\nu s - \tfrac{1}{2}\lambda^2\, |s| + \frac{\lambda^2}{2\beta}\,\Big(1 - e^{-\beta |s|}\Big) \right\} . \tag{3.105}$$

As $\beta \to \infty$, we again recover the result from Eqn. 3.69, and for $\beta = 0$ (which is taken *after* $t \to \infty$), we also obtain the expected result. We see that the coloration of the noise affects the correlator $Y(s)$, resulting in a different time dependence and hence a different spectral function $\hat{Y}(\omega)$.

3.5.4 Remarks about financial markets

Let p be the price of a financial asset, such as a single share of stock. We model the dynamics of $p(t)$ by a stochastic process described by the SDE

$$dp = r(p,t)\, dt + \sqrt{2D(p,t)}\, dW(t) , \tag{3.106}$$

where $r(p,t)$ and $D(p,t)$ represent drift and diffusion terms. We might set $r(p,t) = \mu(t)\, p$, where $\mu(t)$ is the current interest rate being paid by banks. What about diffusion? In the late 1950's, M. Osborne noted that stock prices are approximately log-normally distributed. To model this, we can take $D = \frac{1}{2}\lambda^2 p^2$. Thus, our SDE is

$$dp = \mu\, p\, dt + \lambda\, p\, dW(t) . \tag{3.107}$$

As we shall now see, this will lead to some problematic consequences.

We've solved this equation many times before. Changing variables to $x = \ln p$, we have $dx = \big(\mu - \frac{1}{2}\lambda^2\big)\, dt + \lambda\, dW$, and assuming μ and λ are time-independent, we have

$$p(t) = p(0)\, e^{\mu t}\, e^{-\lambda^2 t/2}\, e^{\lambda W(t)} . \tag{3.108}$$

Averaging, we obtain the moments

$$\big\langle p^n(t)\big\rangle = p^n(0)\, e^{n\mu t}\, e^{n(n-1)\lambda^2 t/2} . \tag{3.109}$$

To appreciate the consequences of this result, let's compute the instantaneous variance,

$$\begin{aligned} \mathsf{Var}\, p(t) &= \big\langle p^2(t)\big\rangle - \big\langle p(t)\big\rangle^2 \\ &= p^2(0)\, e^{2\mu t}\big(e^{\lambda^2 t} - 1\big) . \end{aligned} \tag{3.110}$$

The ratio of the standard deviation to the mean is therefore growing exponentially, and the distribution keeps getting broader *ad infinitum*.

Another way to see what is happening is to examine the associated Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\mu\, \frac{\partial}{\partial p}\big(pP\big) + \tfrac{1}{2}\lambda^2\, \frac{\partial^2}{\partial p^2}\big(p^2 P\big) . \tag{3.111}$$

Let's look for a stationary solution by setting the LHS to zero. We integrate once on p to cancel one power of $\frac{d}{dp}$, and set the associated constant of integration to zero, because $P(p=\infty,t)=0$. This leaves

$$\frac{d}{dp}\big(p^2 P\big) = \frac{2\mu}{\lambda^2}\, pP = \frac{2\mu}{\lambda^2 p}\,(p^2 P)\ . \tag{3.112}$$

The solution is a power law,

$$P(p) = C\, p^{2\mu\lambda^{-2}-2} \quad , \tag{3.113}$$

However, no pure power law distribution is normalizable on the interval $[0,\infty)$, so there is no meaningful steady state for this system. If markets can be modeled by such a stochastic differential equation, then this result is a refutation of Adam Smith's "invisible hand", which is the notion that markets should in time approach some sort of stable equilibrium.

Stochastic variance

A more realistic model is obtained by writing[6]

$$dp = \mu\, p\, dt + \sqrt{v(p,t)}\; p\, dW(t)\ , \tag{3.114}$$

where $v(p,t)$ is strongly nonlinear and nonseparable in p and t. Another approach is to assume the variance $v(t)$ is itself stochastic. We write

$$\begin{aligned} dp &= \mu\, p\, dt + \sqrt{v(t)}\; p\, dW(t) \\ dv &= f(p,v,t)\, dt + g(p,v,t)\,\big[\cos\theta\, dW(t) + \sin\theta\, dY(t)\big]\ , \end{aligned} \tag{3.115}$$

where $W(t)$ and $Y(t)$ are independent Wiener processes. The variance $v(t)$ of stock prices is observed to relax on long-ish time scales of $\gamma^{-1} \approx 22$ days. This is particularly true for aggregate quantities such as market indices (*e.g.* the Dow-Jones Industrial Average (DJIA) or the Deutscher Aktien-Index (DAX)). One typically assumes

$$f(p,v,t) = \gamma(v_\infty - v)\ , \tag{3.116}$$

describing a drift toward $v = v_\infty$, similar to the drift in the Ornstein-Uhlenbeck model. As for the diffusive term $g(p,v,t)$, two popular models are the Heston and Hull-White models:

$$g(p,v,t) = \begin{cases} \kappa\sqrt{v} & \text{Heston} \\ \beta v & \text{Hull-White}\ . \end{cases} \tag{3.117}$$

Empirically, $\theta \approx \frac{\pi}{2}$, which we shall henceforth assume.

The Fokker-Planck equation for the distribution of the variance, $P(v,t)$, is given by

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\big[\gamma(v - v_\infty)\, P\big] + \frac{1}{2}\,\frac{\partial^2}{\partial v^2}\big[g^2(v) P\big]\ . \tag{3.118}$$

We seek a steady state solution for which the LHS vanishes. Assuming $vP(v) \to 0$ for $v \to \infty$, we integrate setting the associated constant of integration to zero. This results in the equation

$$\frac{d}{dv}\Big[g^2(v)\, P(v)\Big] = 2\gamma\bigg(\frac{v_\infty - v}{g^2(v)}\bigg)\, g^2(v)\, P(v)\ , \tag{3.119}$$

[6]See the discussion in McCauley, §4.5 and chapter 6.

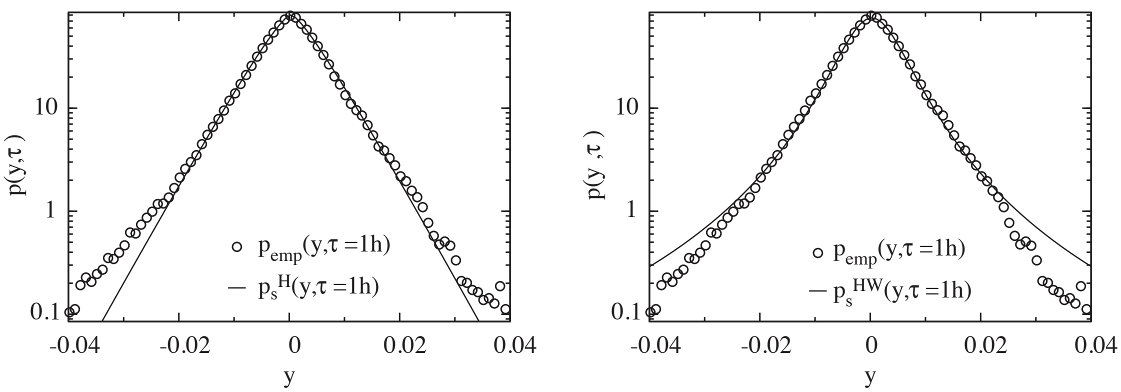

Figure 3.1: Comparison of predictions of the Heston model (left) and the Hull-White model (right) with the empirical probability distribution $P(y,\tau)$ for logarithmic returns of the German DAX index between Feb. 5, 1996 and Dec. 28, 2001 (open circles). Parameters for the Heston model are $r = 1.36$, $v_\infty = 5.15 \times 10^{-5}\,\text{h}^{-1}$, $\mu = 3.03 \times 10^{-4}\,\text{h}^{-1}$. Parameters for the Hull-White model are $s = 0.08$, $v_\infty = 3.21 \times 10^{-4}\,\text{h}^{-1}$, and $\mu = 2.97 \times 10^{-4}\,\text{h}^{-1}$. The time interval was taken to be $\tau = 1\,\text{h}$. From R. Remer and R. Mahnke, *Physica A* **344**, 236 (2004).

with solution

$$P(v) = \frac{1}{g^2(v)}\,\exp\left\{2\gamma\int^v\! dv'\,\left(\frac{v_\infty - v'}{g^2(v')}\right)\right\} . \tag{3.120}$$

For the Heston model, we find

$$P_{\text{H}}(v) = C_{\text{H}}\, v^{(2\gamma v_\infty \kappa^{-2} - 1)}\, e^{-2\gamma v/\gamma^2} , \tag{3.121}$$

whereas for the Hull-White model,

$$P_{\text{HW}}(v) = C_{\text{HW}}\, v^{-2(1+\gamma\beta^{-2})}\, e^{-2\gamma v_\infty/\beta^2 v} . \tag{3.122}$$

Note that both distributions are normalizable. The explicit normalized forms are:

$$\begin{aligned} P_{\text{H}}(v) &= \frac{r^r}{\Gamma(r)\, v_\infty}\left(\frac{v}{v_\infty}\right)^{r-1} \exp\big(-\, rv/v_\infty\big) \\ P_{\text{HW}}(v) &= \frac{s^s}{\Gamma(s)\, v_\infty}\left(\frac{v_\infty}{v}\right)^{s+2} \exp\big(-\, sv_\infty/v\big) , \end{aligned} \tag{3.123}$$

with $r = 2\gamma v_\infty/\kappa^2$ and $s = 2\gamma/\beta^2$. Note that the tails of the Heston model variance distribution are exponential with a power law prefactor, while those of the Hull-White model are power law ”fat tails”.

The SDE for the logarithmic price $x = \ln\big[p(t)/p(0)\big]$, obtained from Itô’s change of variables formula, is

$$dx = \tilde{\mu}\, dt + \sqrt{v}\, dW(t) , \tag{3.124}$$

where $\tilde{\mu} = \mu - \frac{1}{2}\, v$. Here we assume that v is approximately constant in time as $x(t)$ fluctuates. This is akin to the Born-Oppenheimer approximation in quantum mechanics – we regard $v(t)$ as the ”slow variable” and $x(t)$ as the ”fast variable”. Integrating this over a short time interval τ, we have

$$y = \tilde{\mu}\tau + \sqrt{v}\,\Delta W , \tag{3.125}$$

with $y = x(t+\tau) - x(t)$ and $\Delta W = W(t+\tau) - W(t)$. This says that $y - \tilde{\mu}\tau$ is distributed normally with variance $\big\langle(\sqrt{v}\,\Delta W)^2\big\rangle = v\tau$, hence

$$P(y,\tau\,|\,v) = (2\pi v\tau)^{-1/2} \exp\left\{-\frac{(y-\tilde{\mu}\tau)^2}{2v\tau}\right\} . \tag{3.126}$$

To find the distribution $P(y,\tau)$ of the logarithmic returns y, we must integrate over v with a weight $P(v)$, the steady state distribution of the variance:

$$P(y,\tau) = \int_0^\infty dv\; P(y,\tau\,|\,v)\,P(v) . \tag{3.127}$$

The results for the Heston and Hull-White models are shown in Fig. 3.1, where they are compared with empirical data from the DAX.