

Contents

- Contents i

- List of Figures ii

- List of Tables iii

- 1 Elementary Mechanics 1**
 - 1.1 Introduction and Review 1
 - 1.1.1 Newton’s laws of motion 1
 - 1.1.2 Aside : inertial *vs.* gravitational mass 3
 - 1.2 Examples of Motion in One Dimension 4
 - 1.2.1 Uniform force 4
 - 1.2.2 Uniform force with linear frictional damping 5
 - 1.2.3 Uniform force with quadratic frictional damping 6
 - 1.2.4 Crossed electric and magnetic fields 7
 - 1.2.5 Pause for Reflection 8
 - 1.3 Work-Energy Theorem 8
 - 1.4 Conservative and Nonconservative Forces 9
 - 1.4.1 Example : integrating $\mathbf{F} = -\nabla U$ 11
 - 1.4.2 Conservative Fofrces in many-particle systems 12
 - 1.4.3 Linear and angular momentum 13
 - 1.5 Scaling of Solutions for Homogeneous Potentials 15

1.5.1	Euler's theorem for homogeneous functions	15
1.5.2	Scaled equations of motion	15
1.6	Description as a Dynamical System	17
1.6.1	Example : harmonic oscillator	18
1.6.2	One-dimensional mechanics as a dynamical system	18
1.6.3	Sketching phase curves	19
1.6.4	Fixed points and their vicinity	20
1.6.5	Linearized dynamics in the vicinity of a fixed point	21
1.7	Appendix: Examples of Conservative One-Dimensional Systems	22
1.7.1	Harmonic oscillator	22
1.7.2	Pendulum	24
1.7.3	Other potentials	25
1.8	Appendix: Curvilinear Orthogonal Coordinates	29
1.8.1	Example : spherical coordinates	30
1.8.2	Vector calculus : grad, div, curl	30

List of Figures

1.1	Two paths joining points A and B	9
1.2	A potential $U(x)$ and the corresponding phase portraits (with separatrices in red)	20
1.3	Phase curves in the vicinity of centers and saddles	22
1.4	Phase curves for the harmonic oscillator.	23
1.5	Phase curves for the simple pendulum	25
1.6	Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$	26
1.7	Phase curves for the potential $U(x) = -\operatorname{sech}^2(x)$	27
1.8	Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$	28
1.9	Volume element Ω for computing divergences	30

List of Tables

Chapter 1

Elementary Mechanics

1.1 Introduction and Review

Dynamics is the science of how things *move*. A complete solution to the motion of a system means that we know the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, this means we want to know its position vector $\mathbf{r}(t)$ as a function of time. If there are many particles, the motion is described by a set of functions $\mathbf{r}_i(t)$, where i labels which particle we are talking about. So generally speaking, solving for the motion means being able to predict where a particle will be at any given instant of time. Of course, knowing the function $\mathbf{r}_i(t)$ means we can take its derivative and obtain the velocity $\mathbf{v}_i(t) = d\mathbf{r}_i/dt$ at any time as well.

The complete motion for a system is not given to us outright, but rather is encoded in a set of differential equations, called the *equations of motion*. An example of an equation of motion is

$$m \frac{d^2x}{dt^2} = -mg \quad (1.1)$$

with the solution

$$x(t) = x_0 + v_0 t - \frac{1}{2}gt^2 \quad (1.2)$$

where x_0 and v_0 are constants corresponding to the initial *boundary conditions* on the position and velocity: $x(0) = x_0, v(0) = v_0$. This particular solution describes the vertical motion of a particle of mass m moving near the earth's surface.

In this class, we shall discuss a general framework by which the equations of motion may be obtained, and methods for solving them. That "general framework" is Lagrangian Dynamics, which itself is really nothing more than an elegant restatement of Isaac Newton's Laws of Motion.

1.1.1 Newton's laws of motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens.

To paraphrase Wolfgang Pauli, such notions are so vague as to be “not even wrong.” It was only with the publication of Newton’s *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton’s three Laws of Motion may be stated as follows:

- I. A body remains in uniform motion unless acted on by a force.
- II. Force equals rate of change of momentum: $\mathbf{F} = d\mathbf{p}/dt$.
- III. Any two bodies exert equal and opposite forces on each other.

Newton’s First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a “free” particle and that indeed it is in uniform motion, so that $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t$. Now $\mathbf{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\mathbf{r}(t)$ in a different coordinate system whose origin \mathbf{R} moves according to the function $\mathbf{R}(t)$, then in this new “frame of reference” the position of our particle will be

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{r}(t) - \mathbf{R}(t) \\ &= \mathbf{r}_0 + \mathbf{v}_0 t - \mathbf{R}(t) \quad .\end{aligned}\tag{1.3}$$

If the acceleration $d^2\mathbf{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton’s First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton’s Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton’s Laws hold. A transformation from one frame \mathcal{K} to another frame \mathcal{K}' which moves at constant velocity \mathbf{V} relative to \mathcal{K} is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth’s surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth’s rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system’s origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious “forces” such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton’s “quantity of motion” is the momentum \mathbf{p} , defined as the product $\mathbf{p} = m\mathbf{v}$ of a particle’s mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force \mathbf{F} depends on the coordinates (or possibly

velocities) themselves. This is known as a *force law*. Examples of force laws include:

$$\text{Constant force: } \mathbf{F} = -m\mathbf{g}$$

$$\text{Hooke's Law: } F = -kx$$

$$\text{Gravitation: } \mathbf{F} = -GMm\hat{\mathbf{r}}/r^2$$

$$\text{Lorentz force: } \mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B}$$

$$\text{Fluid friction (} v \text{ small): } \mathbf{F} = -b\mathbf{v} \quad .$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\mathbf{F}_1 = -\mathbf{F}_2$ and hence the accelerations are parallel to each other, with

$$\frac{a_1}{a_2} = -\frac{m_2}{m_1} \quad , \quad (1.4)$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ is a constant, a result known as the *conservation of momentum*.

1.1.2 Aside : inertial vs. gravitational mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force \mathbf{F}_{ij} exerted by a particle i by another particle j is

$$\mathbf{F}_{ij} = -Gm_i m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad , \quad (1.5)$$

where G , the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2 \quad . \quad (1.6)$$

Notice Newton's Third Law in action: $\mathbf{F}_{ij} + \mathbf{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_e$, with $\mathbf{r}_i - \mathbf{r}_j \simeq R_e\hat{\mathbf{r}}$ and obtain

$$\mathbf{F} = -mg\hat{\mathbf{r}} \equiv -m\mathbf{g} \quad (1.7)$$

where \hat{r} is a radial unit vector pointing from the earth's center and $g = GM_e/R_e^2 \simeq 9.8 \text{ m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\mathbf{a} = -\mathbf{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m' . In this case, Newton's Second Law would predict

$$\mathbf{a} = -\frac{m'}{m} \mathbf{g} \quad (1.8)$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.2 Examples of Motion in One Dimension

To gain some experience with solving equations of motion in a physical setting, we consider some physically relevant examples of one-dimensional motion.

1.2.1 Uniform force

With $F = -mg$, appropriate for a particle falling under the influence of a uniform gravitational field, we have $m d^2x/dt^2 = -mg$, or $\ddot{x} = -g$. Notation:

$$\dot{x} \equiv \frac{dx}{dt} \quad , \quad \ddot{x} \equiv \frac{d^2x}{dt^2} \quad , \quad \dddot{x} \equiv \frac{d^3x}{dt^3} \quad , \quad \text{etc.} \quad (1.9)$$

With $v = \dot{x}$, we solve $dv/dt = -g$:

$$\int_{v(0)}^{v(t)} dv = \int_0^t ds (-g) \quad (1.10)$$

$$v(t) - v(0) = -gt \quad .$$

Note that there is a constant of integration, $v(0)$, which enters our solution.

We are now in position to solve $dx/dt = v$:

$$\int_{x(0)}^{x(t)} dx = \int_0^t ds v(s) \quad (1.11)$$

$$x(t) = x(0) + \int_0^t ds [v(0) - gs]$$

$$= x(0) + v(0)t - \frac{1}{2}gt^2 \quad .$$

Note that a second constant of integration, $x(0)$, has appeared.

1.2.2 Uniform force with linear frictional damping

In this case,

$$m \frac{dv}{dt} = -mg - \gamma v \quad (1.12)$$

which may be rewritten

$$\begin{aligned} \frac{dv}{v + mg/\gamma} &= -\frac{\gamma}{m} dt \\ d \ln(v + mg/\gamma) &= -(\gamma/m) dt \quad . \end{aligned} \quad (1.13)$$

Integrating then gives

$$\begin{aligned} \ln \left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma} \right) &= -\gamma t/m \\ v(t) &= -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma} \right) e^{-\gamma t/m} \quad . \end{aligned} \quad (1.14)$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$x(t) = x(0) + \frac{m}{\gamma} \left(v(0) + \frac{mg}{\gamma} \right) (1 - e^{-\gamma t/m}) - \frac{mg}{\gamma} t \quad . \quad (1.15)$$

The solution to the *second* order ODE $m\ddot{x} = -mg - \gamma\dot{x}$ thus entails *two* constants of integration: $v(0)$ and $x(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively “forgotten” on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \quad . \quad (1.16)$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity}) \quad ,$$

thus

$$j = n \cdot (-e) \cdot (-v_\infty) = \frac{ne^2\tau}{m} E \quad . \quad (1.17)$$

The ratio j/E is called the *conductivity* of the metal, σ . According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations (“phonons”). In high purity copper at low temperatures ($T \lesssim 4$ K), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}$ s).

1.2.3 Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_f = -cv^2 \operatorname{sgn}(v)$, where $\operatorname{sgn}(v)$ is the *sign* of v : $\operatorname{sgn}(v) = +1$ if $v > 0$ and $\operatorname{sgn}(v) = -1$ if $v < 0$. (Note one can also write $\operatorname{sgn}(v) = v/|v|$ where $|v|$ is the *absolute value*.) Why all this trouble with $\operatorname{sgn}(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be *oppositely directed* with respect to the velocity v . We will assume that $v < 0$ always, hence $F_f = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \quad (1.18)$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty} \left[\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty} \right] \quad (1.19)$$

we obtain

$$\begin{aligned} \frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty} \frac{dv}{v - v_\infty} - \frac{1}{2v_\infty} \frac{dv}{v + v_\infty} \\ &= \frac{1}{2v_\infty} d \ln \left(\frac{v_\infty - v}{v_\infty + v} \right) = \frac{g}{v_\infty^2} dt \quad . \end{aligned} \quad (1.20)$$

Assuming $v(0) = 0$, we integrate to obtain

$$\frac{1}{2v_\infty} \ln \left(\frac{v_\infty - v(t)}{v_\infty + v(t)} \right) = \frac{gt}{v_\infty^2} \quad (1.21)$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \quad . \quad (1.22)$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad . \quad (1.23)$$

Again, as $t \rightarrow \infty$ one has $v(t) \rightarrow -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Advanced Digression: To gain an understanding of the constant c , consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt , all the fluid molecules inside the volume $\Delta V = S \cdot v \Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u in the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\Delta u = 2(v - u)$. The total momentum

change is obtained by multiplying Δu by the total mass $M = \rho \Delta V$, where ρ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v - u) \cdot \rho S v \Delta t \quad (1.24)$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S \rho v(v - u) \quad (1.25)$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u , and since on average $\langle u \rangle = 0$, we obtain the result $\langle F \rangle = 2S \rho v^2$, where $\langle \dots \rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_f = -\langle F \rangle = -c v^2$, where $c = 2S \rho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu \rho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.2.4 Crossed electric and magnetic fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular \mathbf{E} and \mathbf{B} fields. We'll throw in gravity for good measure. We take $\mathbf{E} = E \hat{x}$, $\mathbf{B} = B \hat{z}$, and $\mathbf{g} = -g \hat{z}$. The equation of motion is Newton's 2nd Law again:

$$m \ddot{\mathbf{r}} = m \mathbf{g} + q \mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B} \quad (1.26)$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$\begin{aligned} m \ddot{x} &= qE + \frac{qB}{c} \dot{y} \\ m \ddot{y} &= -\frac{qB}{c} \dot{x} \\ m \ddot{z} &= -mg \end{aligned} \quad (1.27)$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0)t - \frac{1}{2}gt^2 \quad (1.28)$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\begin{aligned} \dot{v}_x &= \omega_c(v_y + u_D) \\ \dot{v}_y &= -\omega_c v_x \end{aligned} \quad (1.29)$$

where $\omega_c = qB/mc$ is the *cyclotron frequency* and $u_D = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$\begin{aligned} v_x(t) &= v_x(0) \cos(\omega_c t) + (v_y(0) + u_D) \sin(\omega_c t) \\ v_y(t) &= -u_D + (v_y(0) + u_D) \cos(\omega_c t) - v_x(0) \sin(\omega_c t) \end{aligned} \quad (1.30)$$

Integrating again, the full motion is given by:

$$\begin{aligned} x(t) &= x(0) + A \sin \delta + A \sin(\omega_c t - \delta) \\ y(r) &= y(0) - u_D t - A \cos \delta + A \cos(\omega_c t - \delta) \quad , \end{aligned} \quad (1.31)$$

where

$$A = \frac{1}{\omega_c} \sqrt{\dot{x}^2(0) + (\dot{y}(0) + u_D)^2} \quad , \quad \delta = \tan^{-1} \left(\frac{\dot{y}(0) + u_D}{\dot{x}(0)} \right) \quad . \quad (1.32)$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0) \quad , \quad y(0) \quad , \quad z(0) \quad , \quad A \quad , \quad \delta \quad , \quad \dot{z}(0) \quad . \quad (1.33)$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

1.2.5 Pause for Reflection

In mechanical systems, for each coordinate, or “degree of freedom,” there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

1.3 Work-Energy Theorem

Consider a system of many particles, with positions \mathbf{r}_i and velocities $\dot{\mathbf{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \quad . \quad (1.34)$$

Now let’s consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i \quad . \quad (1.35)$$

Here, we’ve used the relation

$$\frac{d}{dt} (A^2) = 2 A \cdot \frac{dA}{dt} \quad . \quad (1.36)$$

We now invoke Newton’s 2nd Law, $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$, to write eqn. 1.35 as $\dot{T}_i = \mathbf{F}_i \cdot \dot{\mathbf{r}}_i$. We integrate this equation from time t_A to t_B :

$$T_i^{(B)} - T_i^{(A)} = \int_{t_A}^{t_B} dt \frac{dT_i}{dt} = \int_{t_A}^{t_B} dt \mathbf{F}_i \cdot \dot{\mathbf{r}}_i \equiv \sum_i W_i^{(A \rightarrow B)} \quad , \quad (1.37)$$

where $W_i^{(A \rightarrow B)}$ is the total *work done* on particle i during its motion from state A to state B . Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{(A \rightarrow B)} = \sum_i W_i^{(A \rightarrow B)}$. Eqn. 1.37 is known as the *work-energy theorem*. It says that *In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done*: $T^{(B)} - T^{(A)} = W^{(A \rightarrow B)}$.

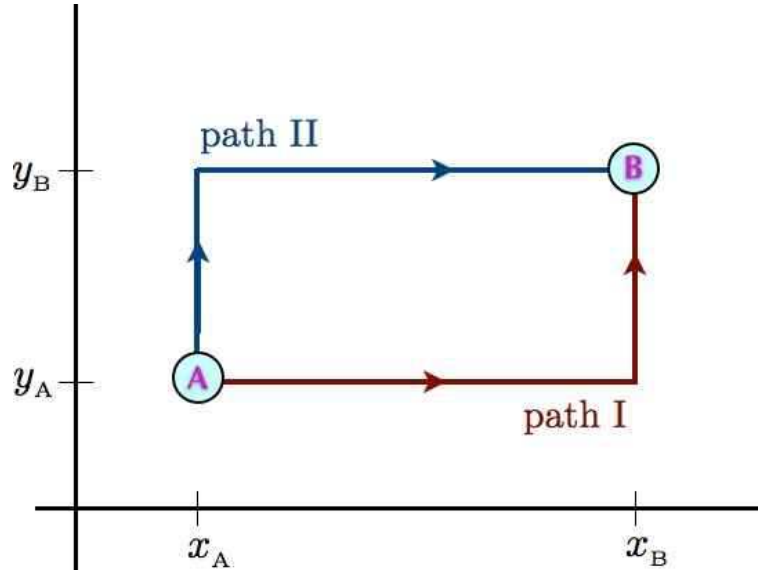


Figure 1.1: Two paths joining points A and B.

1.4 Conservative and Nonconservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\mathbf{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{(A \rightarrow B)} = \int_{t_A}^{t_B} dt \mathbf{F} \cdot \mathbf{v} \quad , \quad (1.38)$$

where $\mathbf{v} = \dot{\mathbf{r}}$. This is the most general expression for the work done. If the force \mathbf{F} depends only on the particle's position \mathbf{r} , we may write $d\mathbf{r} = \mathbf{v} dt$, and then

$$W^{(A \rightarrow B)} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \quad . \quad (1.39)$$

Consider now the force

$$\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}} \quad , \quad (1.40)$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 1.1:

$$\begin{aligned} W^{(I)} &= K_1 \int_{x_A}^{x_B} dx y_A + K_2 \int_{y_A}^{y_B} dy x_B = K_1 y_A (x_B - x_A) + K_2 x_B (y_B - y_A) \\ W^{(II)} &= K_1 \int_{x_A}^{x_B} dx y_B + K_2 \int_{y_A}^{y_B} dy x_A = K_1 y_B (x_B - x_A) + K_2 x_A (y_B - y_A) \quad . \end{aligned} \quad (1.41)$$

Note that in general $W^{(I)} \neq W^{(II)}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(I)} - W^{(II)} = (K_2 - K_1)(x_B - x_A)(y_B - y_A) \quad . \quad (1.42)$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 1.1. The reason for this is Stokes' theorem:

$$\oint_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F} = \int_{\mathcal{C}} dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F} \quad . \quad (1.43)$$

Here, \mathcal{C} is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of \mathcal{C} , which is a closed path¹, dS is the scalar differential area element, $\hat{\mathbf{n}}$ is the unit normal to that differential area element, and $\nabla \times \mathbf{F}$ is the curl of \mathbf{F} :

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \quad . \end{aligned} \quad (1.44)$$

For the force under consideration, $\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}}$, the curl is

$$\nabla \times \mathbf{F} = (K_2 - K_1) \hat{\mathbf{z}} \quad , \quad (1.45)$$

which is a constant. The RHS of eqn. 1.43 is then simply proportional to the area enclosed by \mathcal{C} . When we compute the work difference in eqn. 1.42, we evaluate the integral $\oint_{\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F}$ along the path $\gamma_{II}^{-1} \circ \gamma_I$, which is to say path I followed by the inverse of path II. In this case, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the integral of $\hat{\mathbf{n}} \cdot \nabla \times \mathbf{F}$ over the rectangle \mathcal{C} is given by the RHS of eqn. 1.42.

When $\nabla \times \mathbf{F} = 0$ everywhere in space, we can always write $\mathbf{F} = -\nabla U$, where $U(\mathbf{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$W^{(A \rightarrow B)} = \int_{r_A}^{r_B} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) = - \int_{r_A}^{r_B} d\mathbf{r} \cdot \nabla U = U(\mathbf{r}_A) - U(\mathbf{r}_B) \quad . \quad (1.46)$$

The work-energy theorem then gives

$$T^{(B)} - T^{(A)} = U(\mathbf{r}_A) - U(\mathbf{r}_B) \quad , \quad (1.47)$$

which says

$$E^{(B)} = T^{(B)} + U(\mathbf{r}_B) = T^{(A)} + U(\mathbf{r}_A) = E^{(A)} \quad . \quad (1.48)$$

Thus, the total energy $E = T + U$ is conserved.

¹If \mathcal{C} is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if \mathcal{C} is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

1.4.1 Example : integrating $\mathbf{F} = -\nabla U$

If $\nabla \times \mathbf{F} = 0$, we can compute $U(\mathbf{r})$ by integrating, *viz.*

$$U(\mathbf{r}) = U(\mathbf{0}) - \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}' \cdot \mathbf{F}(\mathbf{r}') \quad . \quad (1.49)$$

The integral does not depend on the path chosen connecting $\mathbf{0}$ and \mathbf{r} . For example, we can take

$$U(x, y, z) = U(0, 0, 0) - \int_{(0,0,0)}^{(x,0,0)} dx' F_x(x', 0, 0) - \int_{(x,0,0)}^{(x,y,0)} dy' F_y(x, y', 0) - \int_{(z,y,0)}^{(x,y,z)} dz' F_z(x, y, z') \quad . \quad (1.50)$$

The constant $U(0, 0, 0)$ is arbitrary and impossible to determine from \mathbf{F} alone.

As an example, consider the force

$$\mathbf{F}(\mathbf{r}) = -ky \hat{\mathbf{x}} - kx \hat{\mathbf{y}} - 4bz^3 \hat{\mathbf{z}} \quad , \quad (1.51)$$

where k and b are constants. We have

$$\begin{aligned} (\nabla \times \mathbf{F})_x &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) = 0 \\ (\nabla \times \mathbf{F})_y &= \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) = 0 \\ (\nabla \times \mathbf{F})_z &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0 \quad , \end{aligned} \quad (1.52)$$

so $\nabla \times \mathbf{F} = 0$ and \mathbf{F} must be expressible as $\mathbf{F} = -\nabla U$. Integrating using eqn. 1.50, we have

$$\begin{aligned} U(x, y, z) &= U(0, 0, 0) + \int_{(0,0,0)}^{(x,0,0)} dx' k \cdot 0 + \int_{(x,0,0)}^{(x,y,0)} dy' kxy' + \int_{(z,y,0)}^{(x,y,z)} dz' 4bz'^3 \\ &= U(0, 0, 0) + kxy + bz^4 \quad . \end{aligned} \quad (1.53)$$

Another approach is to integrate the partial differential equation $\nabla U = -\mathbf{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\mathbf{x}}$ -component,

$$\frac{\partial U}{\partial x} = ky \quad . \quad (1.54)$$

Integrating, we obtain

$$U(x, y, z) = kxy + f(y, z) \quad , \quad (1.55)$$

where $f(y, z)$ is at this point an *arbitrary function* of y and z . The important thing is that it has no x -dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \implies U(x, y, z) = kxy + g(x, z) \quad . \quad (1.56)$$

Finally, the z -component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \implies U(x, y, z) = bz^4 + h(x, y) \quad . \quad (1.57)$$

We now equate the first two expressions:

$$kxy + f(y, z) = kxy + g(x, z) \quad . \quad (1.58)$$

Subtracting kxy from each side, we obtain the equation $f(y, z) = g(x, z)$. Since the LHS is independent of x and the RHS is independent of y , we must have

$$f(y, z) = g(x, z) = q(z) \quad , \quad (1.59)$$

where $q(z)$ is some unknown function of z . But now we invoke the final equation, to obtain

$$bz^4 + h(x, y) = kxy + q(z) \quad . \quad (1.60)$$

The only possible solution is $h(x, y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x, y, z) = C + kxy + bz^4 \quad . \quad (1.61)$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x, y, z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z . The fact that $f(y, z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y, z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y, z)$.

1.4.2 Conservative Forces in many-particle systems

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \\ U &= \sum_i V(\mathbf{r}_i) + \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) \quad . \end{aligned} \quad (1.62)$$

Here, $V(\mathbf{r})$ is the *external* (or one-body) potential, and $v(\mathbf{r} - \mathbf{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{int})} \quad , \quad (1.63)$$

with

$$\begin{aligned} \mathbf{F}_i^{(\text{ext})} &= -\frac{\partial V(\mathbf{r}_i)}{\partial \mathbf{r}_i} \\ \mathbf{F}_i^{(\text{int})} &= -\sum_j \frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\mathbf{r}_i} \equiv \sum_j \mathbf{F}_{ij}^{(\text{int})} . \end{aligned} \quad (1.64)$$

Here, $\mathbf{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j :

$$\mathbf{F}_{ij}^{(\text{int})} = -\frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} = -\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} v'(|\mathbf{r}_i - \mathbf{r}_j|) . \quad (1.65)$$

Note that $\mathbf{F}_{ij}^{(\text{int})} = -\mathbf{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ in which case we may write the interparticle force as

$$\mathbf{F}_{ij}^{(\text{int})} = -\hat{\mathbf{r}}_{ij} v'(r_{ij}) . \quad (1.66)$$

1.4.3 Linear and angular momentum

Consider now the total momentum of the system, $\mathbf{P} = \sum_i \mathbf{p}_i$. Its rate of change is

$$\frac{d\mathbf{P}}{dt} = \sum_i \dot{\mathbf{p}}_i = \sum_i \mathbf{F}_i^{(\text{ext})} + \overbrace{\sum_{i \neq j} \mathbf{F}_{ij}^{(\text{int})}}^{\mathbf{F}_{ij}^{(\text{int})} + \mathbf{F}_{ji}^{(\text{int})} = 0} = \mathbf{F}_{\text{tot}}^{(\text{ext})} , \quad (1.67)$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\begin{aligned} \mathbf{P} &= \sum_i m_i \dot{\mathbf{r}}_i = M \dot{\mathbf{R}} \\ M &= \sum_i m_i \quad (\text{total mass}) \\ \mathbf{R} &= \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \quad (\text{center-of-mass}) . \end{aligned} \quad (1.68)$$

Next, consider the total angular momentum,

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i . \quad (1.69)$$

The rate of change of L is then

$$\begin{aligned}
\frac{dL}{dt} &= \sum_i \{m_i \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i + m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i\} \\
&= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \sum_{i \neq j} \mathbf{r}_i \times \mathbf{F}_{ij}^{(\text{int})} \\
&= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \underbrace{\frac{1}{2} \sum_{i \neq j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{(\text{int})}}_{\mathbf{r}_{ij} \times \mathbf{F}_{ij}^{(\text{int})} = 0} \\
&= \mathbf{N}_{\text{tot}}^{(\text{ext})} \quad .
\end{aligned} \tag{1.70}$$

Finally, it is useful to establish the result

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i (\dot{\mathbf{r}}_i - \dot{\mathbf{R}})^2 \quad , \tag{1.71}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the “work-energy theorem” for conservative systems,

$$\begin{aligned}
0 &= \int_{\text{initial}}^{\text{final}} dE = \int_{\text{initial}}^{\text{final}} dT + \int_{\text{initial}}^{\text{final}} dU \\
&= T^{(\text{B})} - T^{(\text{A})} - \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i \quad ,
\end{aligned} \tag{1.72}$$

which is to say

$$\Delta T = T^{(\text{B})} - T^{(\text{A})} = \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i = -\Delta U \quad . \tag{1.73}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \sum_i V(\mathbf{r}_i) + \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) \quad . \tag{1.74}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i \phi(\mathbf{r}_i) \longrightarrow \int d^3r \rho(\mathbf{r}) \phi(\mathbf{r}) \quad , \tag{1.75}$$

where $\rho(\mathbf{r})$ is the mass density, and $\phi(\mathbf{r})$ is any function.

1.5 Scaling of Solutions for Homogeneous Potentials

1.5.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = \lambda^k U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad . \quad (1.76)$$

Here, k is the *degree of homogeneity* of U . Familiar examples include gravity,

$$U(\mathbf{r}_1, \dots, \mathbf{r}_N) = -G \sum_{i < j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad ; \quad k = -1 \quad , \quad (1.77)$$

and the harmonic oscillator,

$$U(q_1, \dots, q_n) = \frac{1}{2} \sum_{\sigma, \sigma'} V_{\sigma\sigma'} q_\sigma q_{\sigma'} \quad ; \quad k = +2 \quad . \quad (1.78)$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1, \dots, x_n)$ is homogeneous:

$$H(\lambda x_1, \dots, \lambda x_n) = \lambda^k H(x_1, \dots, x_n) \quad . \quad (1.79)$$

Then

$$\left. \frac{d}{d\lambda} \right|_{\lambda=1} H(\lambda x_1, \dots, \lambda x_n) = \sum_{i=1}^n x_i \frac{\partial H}{\partial x_i} = kH \quad . \quad (1.80)$$

1.5.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\mathbf{r}_i = \alpha \tilde{\mathbf{r}}_i \quad , \quad t = \beta \tilde{t} \quad . \quad (1.81)$$

Then

$$\frac{d\mathbf{r}_i}{dt} = \frac{\alpha}{\beta} \frac{d\tilde{\mathbf{r}}_i}{d\tilde{t}} \quad , \quad \frac{d^2\mathbf{r}_i}{dt^2} = \frac{\alpha}{\beta^2} \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} \quad . \quad (1.82)$$

The force \mathbf{F}_i is given by

$$\begin{aligned} \mathbf{F}_i &= -\frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) \\ &= -\frac{\partial}{\partial(\alpha \tilde{\mathbf{r}}_i)} \alpha^k U(\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_N) = \alpha^{k-1} \tilde{\mathbf{F}}_i \quad . \end{aligned} \quad (1.83)$$

Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2} m_i \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} = \alpha^{k-1} \tilde{\mathbf{F}}_i \quad . \quad (1.84)$$

If we choose β such that

We now demand

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \quad , \quad (1.85)$$

then the equation of motion is invariant under the rescaling transformation! This means that if $\mathbf{r}(t)$ is a solution to the equations of motion, then so is $\alpha \mathbf{r}(\alpha^{\frac{1}{2}k-1} t)$. This gives us an entire one-parameter family of solutions, for all real positive α .

If $\mathbf{r}(t)$ is periodic with period T , the $\mathbf{r}_i(t; \alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k} T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \quad . \quad (1.86)$$

Here, $\alpha = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly
Thus

$$[v] = \frac{L}{T} \quad \Rightarrow \quad \frac{v'}{v} = \frac{L'}{L} \Big/ \frac{T'}{T} = \alpha^{\frac{1}{2}k} \quad (1.87)$$

and

$$[E] = \frac{ML^2}{T^2} \quad \Rightarrow \quad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2 \Big/ \left(\frac{T'}{T}\right)^2 = \alpha^k \quad (1.88)$$

and

$$[\mathbf{L}] = \frac{ML^2}{T} \quad \Rightarrow \quad \frac{|\mathbf{L}'|}{|\mathbf{L}|} = \left(\frac{L'}{L}\right)^2 \Big/ \frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \quad . \quad (1.89)$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t; \alpha) = \alpha q_\sigma(t) \quad . \quad (1.90)$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\mathbf{r}(t) \longrightarrow \mathbf{r}(t; \alpha) = \alpha \mathbf{r}(\alpha^{-3/2} t) \quad . \quad (1.91)$$

Thus, $r^3 \propto t^2$, i.e.

$$\left(\frac{L'}{L}\right)^2 = \left(\frac{T'}{T}\right)^2 \quad , \quad (1.92)$$

also known as Kepler's Third Law.

1.6 Description as a Dynamical System

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \quad . \quad (1.93)$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \frac{1}{2}m\dot{x}^2 + U(x) \quad , \quad (1.94)$$

is then conserved. This may be verified explicitly:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left[\frac{1}{2}m\dot{x}^2 + U(x) \right] \\ &= \left[m\ddot{x} + U'(x) \right] \dot{x} = 0 \quad . \end{aligned} \quad (1.95)$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m} \left(E - U(x) \right)} \quad . \quad (1.96)$$

Note that the constant E is a constant of integration. The \pm sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \quad , \quad (1.97)$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E , the motion of the system is bounded by the turning points, and confined to the region(s) $U(x) \leq E$. We can integrate eqn. 1.96 to obtain

$$t(x) - t(x_0) = \pm \sqrt{\frac{m}{2}} \int_{x_0}^x \frac{dx'}{\sqrt{E - U(x')}} \quad . \quad (1.98)$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0 . Since

$$E = E_0 = \frac{1}{2}mv_0^2 + U(x_0) \quad , \quad (1.99)$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0 . Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_{\pm}(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m} \int_{x_{-}(E)}^{x_{+}(E)} \frac{dx'}{\sqrt{E - U(x')}} \quad . \quad (1.100)$$

1.6.1 Example : harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \frac{1}{2}kx^2$, hence

$$\frac{dt}{dx} = \pm \sqrt{\frac{m}{2E - kx^2}} \quad . \quad (1.101)$$

The turning points are $x_{\pm}(E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}} \sin \theta \quad . \quad (1.102)$$

We then find

$$dt = \sqrt{\frac{m}{k}} d\theta \quad , \quad (1.103)$$

with solution

$$\theta(t) = \theta_0 + \omega t \quad , \quad (1.104)$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the complete motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \sin(\omega t + \theta_0) \quad . \quad (1.105)$$

Note the two constants of integration, E and θ_0 .

1.6.2 One-dimensional mechanics as a dynamical system

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m} F(x) \quad . \end{aligned} \quad (1.106)$$

This may be written in matrix-vector form, as

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m} F(x) \end{pmatrix} \quad . \quad (1.107)$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\varphi}{dt} = \mathbf{V}(\varphi) \quad , \quad (1.108)$$

where $\varphi = (\varphi_1, \dots, \varphi_N)$ is an N -dimensional vector in *phase space*. For the model of eqn. 1.107, we evidently have $N = 2$. The object $\mathbf{V}(\varphi)$ is called a *vector field*. It is itself a vector, existing at every point

in phase space, \mathbb{R}^N . Each of the components of $\mathbf{V}(\varphi)$ is a function (in general) of *all* the components of φ :

$$V_j = V_j(\varphi_1, \dots, \varphi_N) \quad (j = 1, \dots, N) \quad . \quad (1.109)$$

Solutions to the equation $\dot{\varphi} = \mathbf{V}(\varphi)$ are called *integral curves*. Each such integral curve $\varphi(t)$ is uniquely determined by N constants of integration, which may be taken to be the initial value $\varphi(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm \sqrt{\frac{2}{m}(E - U(x))}$. The phase curves are thus curves of constant energy.

1.6.3 Sketching phase curves

To plot the phase curves,

(i) Sketch the potential $U(x)$.

(ii) Below this plot, sketch $v(x; E) = \pm \sqrt{\frac{2}{m}(E - U(x))}$.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

(a) For E slightly above E_{\min} , the phase curves are ellipses.

(b) For E slightly below E_{\max} , the phase curves are (locally) hyperbolae.

(c) For $E = E_{\max}$ the phase curve is called a *separatrix*.

(iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area \mathcal{A} in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint_E v dx = \sqrt{\frac{8}{m}} \int_{x_-(E)}^{x_+(E)} dx' \sqrt{E - U(x')} \quad . \quad (1.110)$$

Thus, the period is proportional to the rate of change of $\mathcal{A}(E)$ with E :

$$T = m \frac{\partial \mathcal{A}}{\partial E} \quad . \quad (1.111)$$

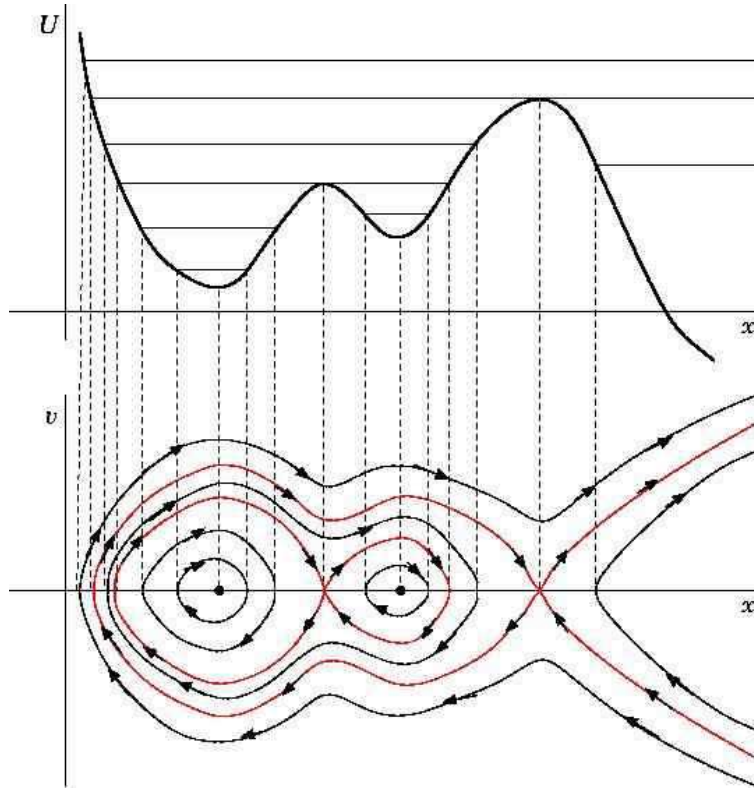


Figure 1.2: A potential $U(x)$ and the corresponding phase portraits (with separatrices in red).

1.6.4 Fixed points and their vicinity

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^* :

$$U(x) = U(x^*) + U'(x^*)(x - x^*) + \frac{1}{2}U''(x^*)(x - x^*)^2 + \frac{1}{6}U'''(x^*)(x - x^*)^3 + \dots \quad (1.112)$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \frac{1}{2}k(\delta x)^2 \quad , \quad (1.113)$$

where $U_0 = U(x^*)$ and $k = U''(x^*) > 0$. The solutions to the motion in this potential are:

$$U''(x^*) > 0 : \delta x(t) = \delta x_0 \cos(\omega t) + \frac{\delta v_0}{\omega} \sin(\omega t) \quad (1.114)$$

$$U''(x^*) < 0 : \delta x(t) = \delta x_0 \cosh(\gamma t) + \frac{\delta v_0}{\gamma} \sinh(\gamma t) \quad ,$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \frac{1}{2}m(\delta v_0)^2 + \frac{1}{2}k(\delta x_0)^2 \quad . \quad (1.115)$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm \gamma \delta x_0$. Let's take $\delta v_0 = -\gamma \delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0 \exp(-\gamma t) \quad . \quad (1.116)$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \ln \left(\frac{\delta x_0}{\delta x} \right) \quad . \quad (1.117)$$

This diverges logarithmically as $\delta x \rightarrow 0$. Generically, then, *the period of motion along a separatrix is infinite.*

1.6.5 Linearized dynamics in the vicinity of a fixed point

Linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{m} U''(x^*) & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} + \dots, \quad (1.118)$$

This is a *linear* equation, which we can solve completely.

Consider the general linear equation $\dot{\varphi} = A \varphi$, where A is a fixed real matrix. Now whenever we have a problem involving matrices, we should start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$A \psi_\alpha = \lambda_\alpha \psi_\alpha \quad . \quad (1.119)$$

Here ψ_α is the α^{th} *right eigenvector*² of A . The eigenvalues are roots of the characteristic equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - A)$. Let's expand $\varphi(t)$ in terms of the right eigenvectors of A :

$$\varphi(t) = \sum_{\alpha} C_{\alpha}(t) \psi_{\alpha} \quad . \quad (1.120)$$

Assuming, for the purposes of this discussion, that A is nondegenerate, and its eigenvectors span \mathbb{R}^N , the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_{\alpha}(t)$:

$$\dot{C}_{\alpha} = \lambda_{\alpha} C_{\alpha} \quad , \quad (1.121)$$

with solutions

$$C_{\alpha}(t) = C_{\alpha}(0) \exp(\lambda_{\alpha} t) \quad . \quad (1.122)$$

If $\text{Re}(\lambda_{\alpha}) > 0$, $C_{\alpha}(t)$ flows off to infinity, while if $\text{Re}(\lambda_{\alpha}) < 0$, $C_{\alpha}(t)$ flows to zero. If $|\lambda_{\alpha}| = 1$, then $C_{\alpha}(t)$ oscillates with frequency $\text{Im}(\lambda_{\alpha})$.

²If A is symmetric, the right and left eigenvectors are the same. If A is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same.

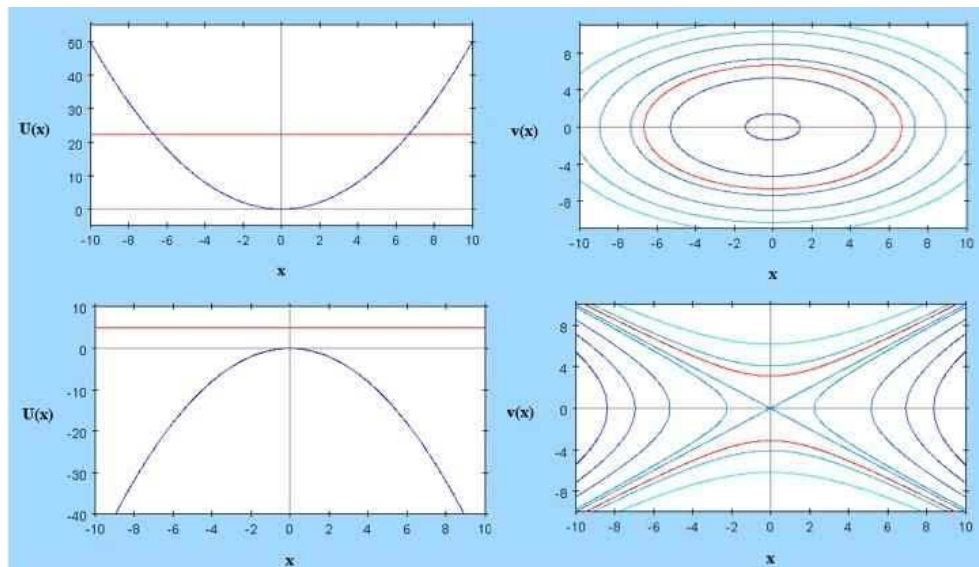


Figure 1.3: Phase curves in the vicinity of centers and saddles.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \quad , \quad (1.123)$$

where $T = \text{Tr}(A)$ and $D = \det(A)$. The eigenvalues are then

$$\lambda_{\pm} = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4D} \quad . \quad (1.124)$$

We'll study the general case in Physics 110B. For now, we focus on our conservative mechanical system of eqn. 1.118. The trace and determinant of the above matrix are $T = 0$ and $D = \frac{1}{m}U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in fig. 1.2.

1.7 Appendix: Examples of Conservative One-Dimensional Systems

1.7.1 Harmonic oscillator

Recall again the harmonic oscillator, discussed in lecture 3. The potential energy is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m \frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \quad , \quad (1.125)$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} \quad , \quad (1.126)$$

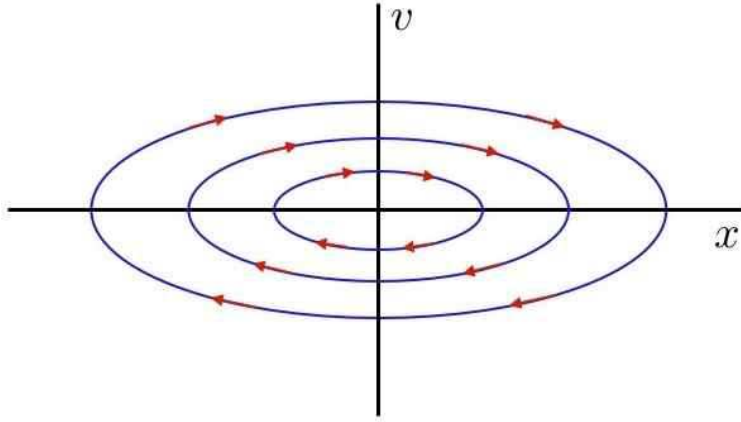


Figure 1.4: Phase curves for the harmonic oscillator.

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$\begin{aligned} x(t) &= x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \\ v(t) &= v_0 \cos(\omega t) - \omega x_0 \sin(\omega t) \end{aligned} \quad (1.127)$$

The phase curves are ellipses:

$$\omega_0 x^2(t) + \omega_0^{-1} v^2(t) = C \quad , \quad (1.128)$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in fig. 1.4. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad . \quad (1.129)$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\max} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\max} = \sqrt{\frac{2E}{m}} \quad . \quad (1.130)$$

The area of the elliptical phase curves is thus

$$A(E) = \pi x_{\max} v_{\max} = \frac{2\pi E}{\sqrt{mk}} \quad . \quad (1.131)$$

The period of motion is therefore

$$T(E) = m \frac{\partial A}{\partial E} = 2\pi \sqrt{\frac{m}{k}} \quad , \quad (1.132)$$

which is independent of E .

1.7.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ . The potential is $U(\theta) = -mgl \cos \theta$, hence

$$m\ell^2 \ddot{\theta} = -\frac{dU}{d\theta} = -mgl \sin \theta \quad . \quad (1.133)$$

This is equivalent to

$$\frac{d}{dt} \begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega_0^2 \sin \theta \end{pmatrix} \quad , \quad (1.134)$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \frac{1}{2} m\ell^2 \dot{\theta}^2 + U(\theta) \quad . \quad (1.135)$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot{\theta}^2 - 2\omega_0^2 \cos \theta = -2\omega_0^2 \cos \theta_0 \quad . \quad (1.136)$$

The period for motion of amplitude θ_0 is then

$$T(\theta_0) = \frac{\sqrt{8}}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \frac{4}{\omega_0} K(\sin^2 \frac{1}{2}\theta_0) \quad , \quad (1.137)$$

where $K(z)$ is the complete elliptic integral of the first kind. Expanding $K(z)$, we have

$$T(\theta_0) = \frac{2\pi}{\omega_0} \left\{ 1 + \frac{1}{4} \sin^2 \left(\frac{1}{2}\theta_0 \right) + \frac{9}{64} \sin^4 \left(\frac{1}{2}\theta_0 \right) + \dots \right\} \quad . \quad (1.138)$$

For $\theta_0 \rightarrow 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2 \theta$. As $\theta_0 \rightarrow \pi/2$, the period diverges logarithmically.

The phase curves for the pendulum are shown in fig. 1.5. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin \theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

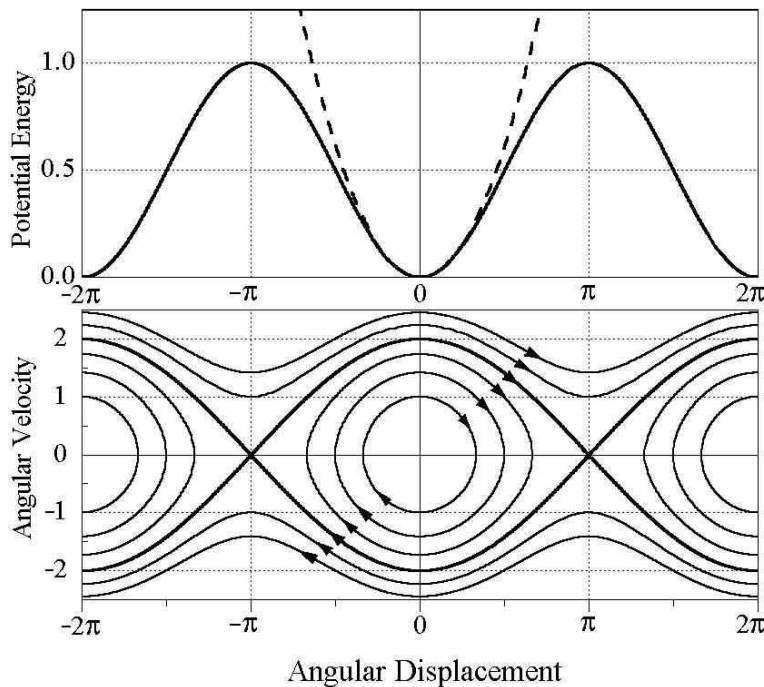


Figure 1.5: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

1.7.3 Other potentials

Using the phase plotter application written by Ben Schmidel, available on the Physics 110A course web page, it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \quad , \quad (1.139)$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses $m_{1,2}$, $\mu = m_1 m_2 / (m_1 + m_2)$ is the 'reduced mass', and $k = G m_1 m_2$, where G is the Cavendish constant. We can then write

$$U_{\text{eff}}(r) = U_0 \left\{ -\frac{1}{x} + \frac{1}{2x^2} \right\} \quad , \quad (1.140)$$

where $r_0 = \ell^2 / \mu k$ has the dimensions of length, and $x \equiv r / r_0$, and where $U_0 = k / r_0 = \mu k^2 / \ell^2$. Thus, if distances are measured in units of r_0 and the potential in units of U_0 , the potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \operatorname{sech}^2(x/a) \quad , \quad (1.141)$$

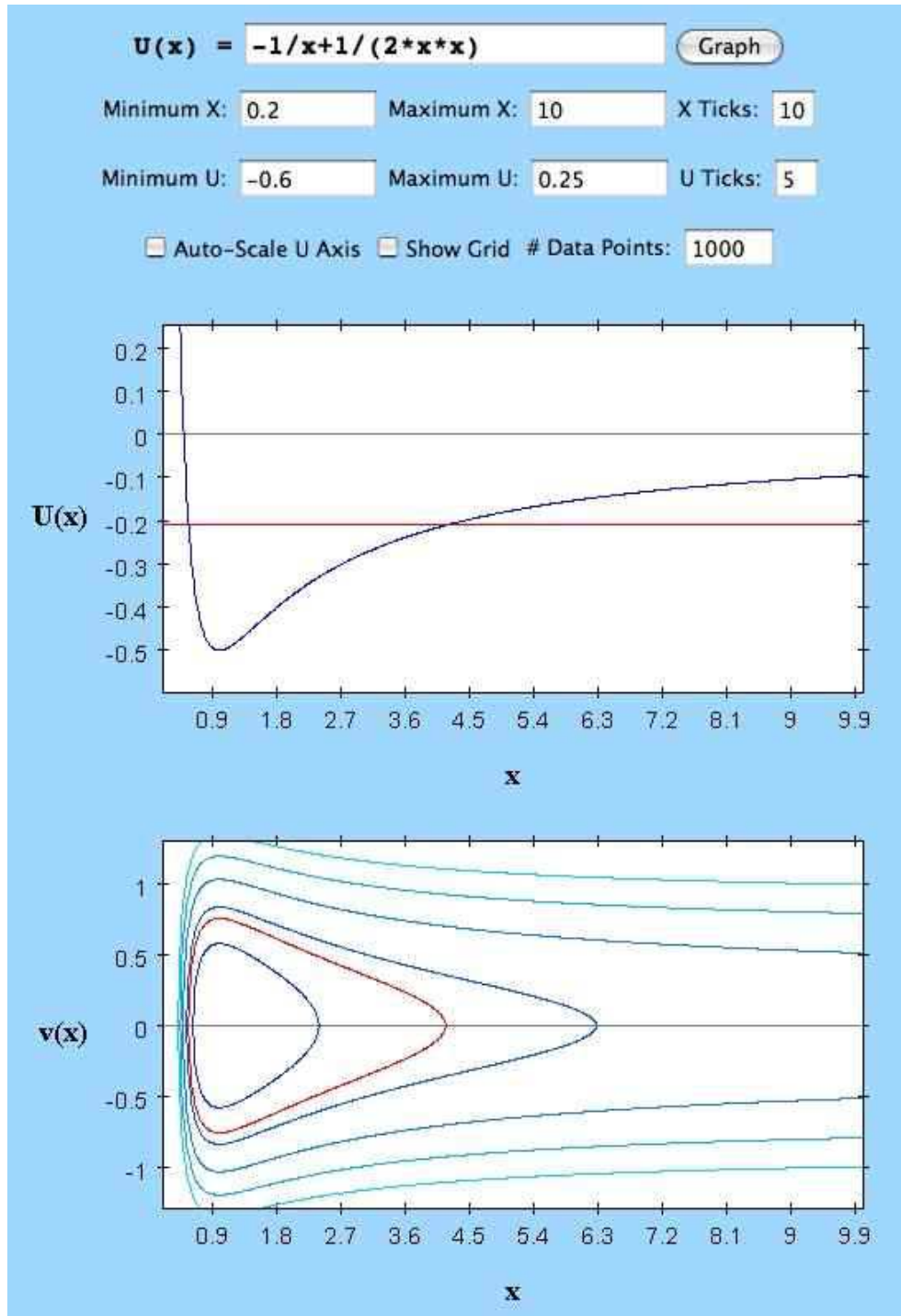
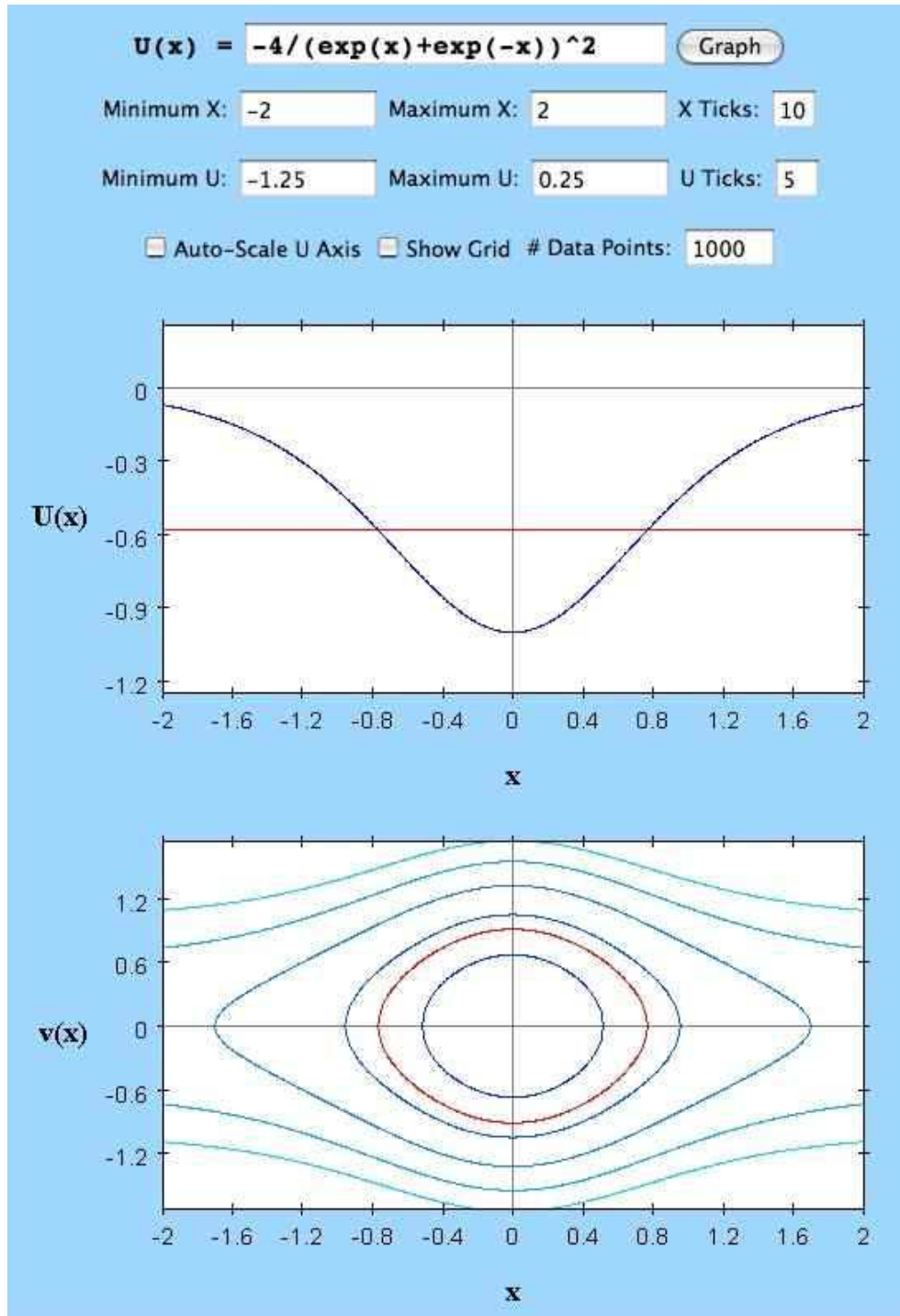


Figure 1.6: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

which, in dimensionless form, is $U(x) = -\text{sech}^2(x)$, after measuring distances in units of a and potential in units of U_0 .

Figure 1.7: Phase curves for the potential $U(x) = -\operatorname{sech}^2(x)$.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} . \quad (1.142)$$

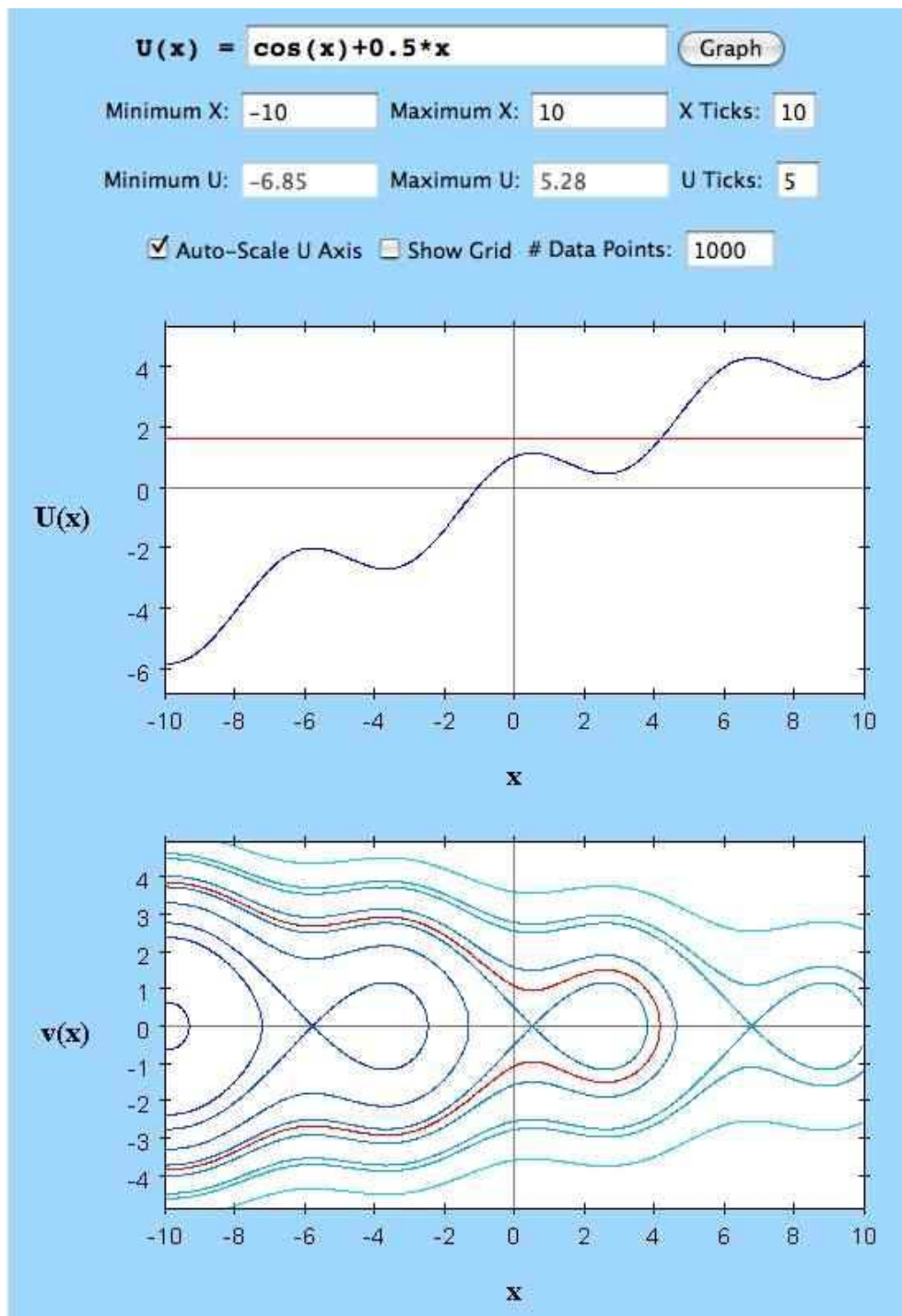


Figure 1.8: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.

Again measuring x in units of a and U in units of U_0 , we arrive at $U(x) = \cos(x) + \frac{1}{2}x$.

1.8 Appendix: Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \dots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \dots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu(x_1, \dots, x_d) \quad . \quad (1.143)$$

In general these are nonlinear equations.

Let $\hat{e}_i^0 = \hat{x}_i$ be the Cartesian set of orthonormal unit vectors, and define \hat{e}_μ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\mathbf{s} = \sum_{i=1}^d \hat{e}_i^0 dx_i = \sum_{\mu=1}^d \hat{e}_\mu h_\mu(q) dq_\mu \quad , \quad (1.144)$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν . Finding the coefficient of dq_μ then gives

$$h_\mu(q) \hat{e}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \hat{e}_i^0 \quad \Rightarrow \quad \hat{e}_\mu = \sum_{i=1}^d M_{\mu i} \hat{e}_i^0 \quad , \quad (1.145)$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \frac{\partial x_i}{\partial q_\mu} \quad . \quad (1.146)$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{e}_\mu \cdot \hat{e}_\nu = (MM^t)_{\mu\nu} = \frac{1}{h_\mu(q) h_\nu(q)} \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} \quad . \quad (1.147)$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q) \delta_{\mu\nu} \quad . \quad (1.148)$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \left(\frac{\partial x_i}{\partial q_\mu} \right)^2} \quad . \quad (1.149)$$

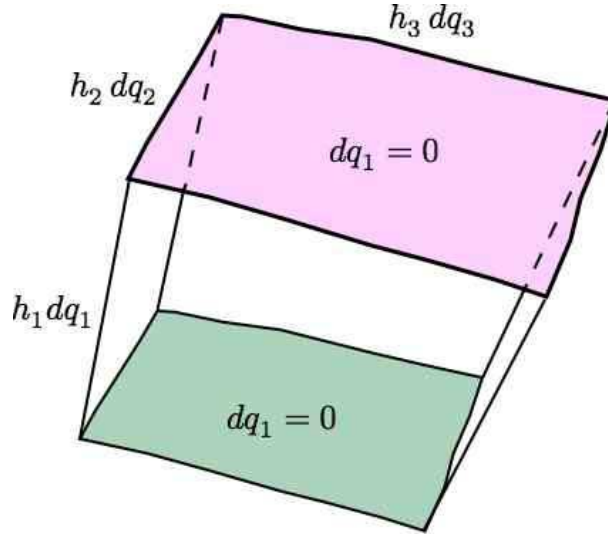
Note that

$$(ds)^2 = \sum_{\mu=1}^d h_\mu^2(q) (dq_\mu)^2 \quad . \quad (1.150)$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(ds)^2 = \sum_{\mu, \nu=1}^d g_{\mu\nu}(q) dq_\mu dq_\nu \quad , \quad (1.151)$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

Figure 1.9: Volume element Ω for computing divergences.

1.8.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ) :

$$x = \rho \sin \theta \cos \phi \quad , \quad y = \rho \sin \theta \sin \phi \quad , \quad z = \rho \cos \theta \quad . \quad (1.152)$$

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2 \sin^2 \theta \quad . \quad (1.153)$$

Thus,

$$ds = \hat{\rho} d\rho + \rho \hat{\theta} d\theta + \rho \sin \theta \hat{\phi} d\phi \quad . \quad (1.154)$$

1.8.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient ∇U of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1} dq_1 + \frac{\partial U}{\partial q_2} dq_2 + \frac{\partial U}{\partial q_3} dq_3 \\ &\equiv \nabla U \cdot ds \quad . \end{aligned} \quad (1.155)$$

Thus,

$$\nabla = \frac{\hat{e}_1}{h_1(q)} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2(q)} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3(q)} \frac{\partial}{\partial q_3} \quad . \quad (1.156)$$

For the divergence, we use the divergence theorem, and we appeal to fig. 1.9:

$$\int_{\Omega} dV \nabla \cdot \mathbf{A} = \int_{\partial\Omega} dS \hat{\mathbf{n}} \cdot \mathbf{A} \quad , \quad (1.157)$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \nabla \cdot \mathbf{A} \cdot (h_1 dq_1) (h_2 dq_2) (h_3 dq_3) \quad . \quad (1.158)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_1 h_2 h_3 \Big|_{q_1}^{q_1+dq_1} dq_2 dq_3 + A_2 h_1 h_3 \Big|_{q_2}^{q_2+dq_2} dq_1 dq_3 + A_3 h_1 h_2 \Big|_{q_3}^{q_3+dq_3} dq_1 dq_2 \\ &= \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] dq_1 dq_2 dq_3 \quad . \end{aligned} \quad (1.159)$$

We therefore conclude

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] \quad . \quad (1.160)$$

To obtain the curl $\nabla \times \mathbf{A}$, we use Stokes' theorem again,

$$\int_{\Sigma} dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_{\partial\Sigma} d\ell \cdot \mathbf{A} \quad , \quad (1.161)$$

where Σ is a two-dimensional region of space and $\partial\Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2 dq_2$ and $h_3 dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\mathbf{e}}_1 \cdot \nabla \times \mathbf{A} (h_2 dq_2) (h_3 dq_3) \quad . \quad (1.162)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_3 h_3 \Big|_{q_2}^{q_2+dq_2} dq_3 - A_2 h_2 \Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \left[\frac{\partial}{\partial q_2} (A_3 h_3) - \frac{\partial}{\partial q_3} (A_2 h_2) \right] dq_2 dq_3 \quad . \end{aligned} \quad (1.163)$$

Therefore

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left(\frac{\partial(h_3 A_3)}{\partial q_2} - \frac{\partial(h_2 A_2)}{\partial q_3} \right) \quad . \quad (1.164)$$

This is one component of the full result

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \det \begin{pmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{pmatrix} \quad . \quad (1.165)$$

The Laplacian of a scalar function U is given by

$$\begin{aligned} \nabla^2 U &= \nabla \cdot \nabla U \\ &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial U}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial U}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial U}{\partial q_3} \right) \right\} \quad . \end{aligned} \quad (1.166)$$

Rectangular coordinates

In *rectangular* coordinates (x, y, z) , we have

$$h_x = h_y = h_z = 1 \quad . \quad (1.167)$$

Thus

$$ds = \hat{x} dx + \hat{y} dy + \hat{z} dz \quad (1.168)$$

and the velocity squared is

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \quad . \quad (1.169)$$

The gradient is

$$\nabla U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z} \quad . \quad (1.170)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad . \quad (1.171)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \quad . \quad (1.172)$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad . \quad (1.173)$$

Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z) , we have

$$\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi \quad , \quad \hat{x} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \quad , \quad d\hat{\rho} = \hat{\phi} d\phi \quad (1.174)$$

and

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \quad , \quad \hat{y} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \quad , \quad d\hat{\phi} = -\hat{\rho} d\phi \quad . \quad (1.175)$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \quad . \quad (1.176)$$

Thus

$$ds = \hat{\rho} d\rho + \hat{\phi} \rho d\phi + \hat{z} dz \quad (1.177)$$

and the velocity squared is

$$\dot{s}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \quad . \quad (1.178)$$

The gradient is

$$\nabla U = \hat{\rho} \frac{\partial U}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial U}{\partial \phi} + \hat{z} \frac{\partial U}{\partial z} \quad . \quad (1.179)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad . \quad (1.180)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} + \left(\frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \hat{z} \quad . \quad (1.181)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \quad . \quad (1.182)$$

Spherical coordinates

In *spherical* coordinates (r, θ, ϕ) , we have

$$\begin{aligned} \hat{r} &= \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \\ \hat{\theta} &= \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta \\ \hat{\phi} &= -\hat{x} \sin \phi + \hat{y} \cos \phi \quad , \end{aligned} \quad (1.183)$$

for which

$$\hat{r} \times \hat{\theta} = \hat{\phi} \quad , \quad \hat{\theta} \times \hat{\phi} = \hat{r} \quad , \quad \hat{\phi} \times \hat{r} = \hat{\theta} \quad . \quad (1.184)$$

The inverse is

$$\begin{aligned} \hat{x} &= \hat{r} \sin \theta \cos \phi + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi \\ \hat{y} &= \hat{r} \sin \theta \sin \phi + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi \\ \hat{z} &= \hat{r} \cos \theta - \hat{\theta} \sin \theta \quad . \end{aligned} \quad (1.185)$$

The differential relations are

$$\begin{aligned} d\hat{r} &= \hat{\theta} d\theta + \sin \theta \hat{\phi} d\phi \\ d\hat{\theta} &= -\hat{r} d\theta + \cos \theta \hat{\phi} d\phi \\ d\hat{\phi} &= -(\sin \theta \hat{r} + \cos \theta \hat{\theta}) d\phi \end{aligned} \quad (1.186)$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r \sin \theta \quad . \quad (1.187)$$

Thus

$$ds = \hat{r} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin \theta d\phi \quad (1.188)$$

and the velocity squared is

$$\dot{s}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \quad . \quad (1.189)$$

The gradient is

$$\nabla U = \hat{\mathbf{r}} \frac{\partial U}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \frac{\partial U}{\partial \phi} \quad . \quad (1.190)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad . \quad (1.191)$$

The curl is

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r \sin \theta} \left(\frac{\partial(\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} \\ & + \frac{1}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \quad . \end{aligned} \quad (1.192)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} \quad . \quad (1.193)$$

Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \frac{1}{2} m \left(\frac{d\mathbf{s}}{dt} \right)^2 &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) && \text{(3D Cartesian)} \\ &= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) && \text{(2D polar)} \\ &= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) && \text{(3D cylindrical)} \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) && \text{(3D polar)} \quad . \end{aligned} \quad (1.194)$$