A point particle of mass $m$ in two-dimensions moves along a one-dimensional surface under the influence of gravity $g = -g \hat{y}$. The equation of the surface is

$$y = x - \frac{x^3}{3a^2}.$$ 

The particle is released from rest at a point along the curve $(x_0, y_0)$. The particle flies off the curve at $x = a$. Determine $y_0$.

**Solution:**

The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

and there is a single constraint,

$$G(x, y) = y - h(x) = 0.$$ 

The Euler-Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \lambda \frac{\partial G}{\partial q_i}.$$ 

Thus, we obtain

$$m\ddot{x} = -\lambda h'(x)$$
$$m\ddot{y} = \lambda - mg.$$ 

The constraint is $y = h(x)$. The constraint may be differentiated to yield

$$\dot{y} = h'(x) \dot{x} \quad , \quad \ddot{y} = h''(x) \dot{x}^2 + h'(x) \ddot{x}.$$ 

Substituting into the second equation of motion, we obtain

$$\frac{\lambda}{m} = g + h''(x) \dot{x}^2 + h'(x) \ddot{x}$$

and thus, from the first equation of motion,

$$(1 + h'(x)^2) \ddot{x} + h'(x) h''(x) \dot{x}^2 = -g h'(x).$$

**NOTE:** This equation of motion is also obtained by eliminating the holonomic constraint $y = h(x)$ at the outset, and writing

$$L = \frac{1}{2}m(1 + h'(x)^2) \dot{x}^2 - mg h(x).$$
The particle flies off the curve when the vertical force of constraint $\lambda$ starts to become negative, because the curve can supply only a positive normal force. To evaluate $y_0$, we must express $\lambda$ in terms of $y_0$ and $x$. Therefore, we must eliminate $\ddot{x}$ and $\dot{x}$ from

$$\frac{\lambda}{m} = g + h''(x) \dot{x}^2 + h'(x) \ddot{x}.$$ 

To eliminate $\ddot{x}$, we can use the equation of motion. This gives

$$\ddot{x} = - \left( \frac{g + h'' \dot{x}^2}{1 + h'^2} \right) h',$$

and thus

$$\lambda = m \left( \frac{g + h'' \dot{x}^2}{1 + h'^2} \right).$$

This has a simple interpretation at points where $h' = 0$: $\lambda = mg + mv^2/R$, where $R$ is the local radius of curvature.

To eliminate $\dot{x}$, using conservation of energy,

$$E = mgy_0 = \frac{1}{2} m \left( 1 + h'^2 \right) \dot{x}^2 + mg h(x),$$

yielding

$$\dot{x}^2 = 2g \frac{y_0 - h}{1 + h'^2}.$$ 

Putting it all together,

$$\lambda = \frac{mg}{(1 + h'^2)^2} \left\{ 1 + h'^2 + 2(y_0 - h) h'' \right\}.$$ 

The particle flies off when $\lambda = 0$. This means

$$1 + h'(x)^2 + 2(y_0 - h(x)) h''(x) = 0.$$

For $h(x) = x - \frac{x^3}{3a^2}$, one obtains $y_0 = \frac{11}{12} a$.

---

[2] Consider the two coupled strings of fig. 1. Both strings are described by identical mass density $\sigma$ and tension $\tau$. On each string, at $x = 0$, a point mass $m$ is affixed. The two masses are connected via a spring of constant $\kappa$. When the two masses are identically displaced, i.e. when $u_1(0,t) = u_2(0,t)$, the spring is unstretched. There are no other forces aside from the tension in the strings and the restoring force of the spring.

(a) Let $u_i(x,t)$ be the displacement field of each string ($i = 1, 2$). Write down the equations of motion for the two masses.
Figure 1: Two identical strings with masses $m$ at $x = 0$, coupled via a spring of constant $\kappa$.

(b) Taking advantage of the symmetry under interchange of the two strings, define sum and difference fields,

$u_{\pm}(x, t) \equiv u_1(x, t) \pm u_2(x, t)$,

and rewrite the equations from part (a) in terms of $u_{\pm}(0, t)$.

(c) In the distant past, a pulse of shape $f(\xi)$ is incident from the left on string #1. Given the definition of the functions $g_i(\xi)$ and $h_i(\xi)$ implicit in the figure, find the complex reflection and transmission coefficients,

$r(k) = \frac{\hat{g}_1(k)}{\hat{f}(k)}$, $t(k) = \frac{\hat{h}_1(k)}{\hat{f}(k)}$, $\tilde{t}(k) = \frac{\hat{g}_2(k)}{\hat{f}(k)}$, $\tilde{r}(k) = \frac{\hat{h}_2(k)}{\hat{f}(k)}$,

where $\hat{f}(k)$ is the Fourier transform of $f(\xi)$, etc.\(^1\) For notational convenience, you should define $Q = \tau/mc^2$ and $P = \sqrt{2\kappa/mc^2}$.

(d) Do your answers make sense in the limits $m \to \infty$ and $\kappa \to 0$?

(e) Consider the limit $m \to 0$ with $\kappa$, $\tau$, and $\sigma$ held fixed. Find the transmission and reflection coefficients $T = |t|^2$, $R = |r|^2$, $\tilde{T} = |\tilde{t}|^2$, $\tilde{R} = |\tilde{r}|^2$, and show that energy flux is conserved.

(f) Write down the Lagrangian density $\mathcal{L}$ for this system.

**Solution:**

(a) For each mass, we write $F = ma$:

$m\ddot{u}_1(0, t) = \tau u_1'(0^+, t) - \tau u_1'(0^-, t) - \kappa[u_1(0, t) - u_2(0, t)]$

$m\ddot{u}_2(0, t) = \tau u_2'(0^+, t) - \tau u_2'(0^-, t) - \kappa[u_2(0, t) - u_1(0, t)]$.

\(^1\) Even though the $g_2$ wave moves to the left, we consider this transmission from one branch of string to another, rather than reflection into the same branch.
Taking sums and differences of the above equations, we have
\[ m\ddot{u}_+(0,t) = \tau u'_+(0^+,t) - \tau u'_+(0^-,t) \]
\[ m\ddot{u}_-(0,t) = \tau u'_-(0^+,t) - \tau u'_-(0^-,t) - 2\kappa u_-(0,t) . \]

Note that the spring does not affect \( u_+(x,t) \)! This is because in the symmetric (+) mode, the masses move together, and the spring remains unstretched, generating no restoring force.

(c) We write
\[ x < 0 : \quad u_\pm(x,t) = f_\pm(ct - x) + g_\pm(ct + x) \]
\[ x > 0 : \quad u_\pm(x,t) = h_\pm(ct - x) , \]
with \( f_\pm = f_1 \pm f_2 \), \( g_\pm = g_1 \pm g_2 \), and \( h_\pm = h_1 \pm h_2 \) (of course). Note that continuity of \( x_\pm(x,t) \) at \( x = 0 \) for all time implies
\[ f_\pm(\xi) + g_\pm(\xi) = h_\pm(\xi) , \]
for all \( \xi \). The results of part (b) may now be written as
\[ mc^2 h''_+(\xi) = \tau f'_+(\xi) - \tau g'_+(\xi) - \tau h'_+(\xi) \]
\[ mc^2 h''_-(\xi) = \tau f'_-(\xi) - \tau g'_-(\xi) - \tau h'_-(\xi) - 2\kappa h_-(\xi) . \]

We next Fourier transform, with
\[ f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} , \quad \hat{f}(k) = \int_{-\infty}^{\infty} d\xi f(\xi) e^{-ik\xi} , \]
et cetera. We obtain four linear, algebraic equations. The first two arise from continuity:
\[ \hat{f}_+(k) + \hat{g}_+(k) = \hat{h}_+(k) , \quad \hat{f}_-(k) + \hat{g}_-(k) = \hat{h}_-(k) . \]
The second two are restatements of \( F = ma \) for each mass. Using continuity to first eliminate \( g_\pm \) in terms of \( f_\pm \) and \( h_\pm \), we obtain, after some rearrangement,
\[ \hat{h}_+(k) = -\frac{2iQ}{k^2 - 2iQ} \hat{f}_+(k) \quad , \quad \hat{h}_-(k) = -\frac{2iQk}{k^2 - 2iQk - P^2} \hat{f}_-(k) , \]
with \( Q = \tau/mc^2 = \sigma/m \) and \( P^2 = 2\kappa/mc^2 = 2\sigma\kappa/\tau m \). Using
\[ f = f_1 = \frac{1}{2} f_+ + \frac{1}{2} f_- \quad , \quad g_1 = \frac{1}{2} g_+ + \frac{1}{2} g_- \quad , \quad h_1 = \frac{1}{2} h_+ + \frac{1}{2} h_- \]
\[ 0 = f_2 = \frac{1}{2} f_+ - \frac{1}{2} f_- \quad , \quad g_2 = \frac{1}{2} g_+ - \frac{1}{2} g_- \quad , \quad h_2 = \frac{1}{2} h_+ - \frac{1}{2} h_- , \]
we find
\[ t(k) = \frac{\dot{h}_1(k)}{\dot{f}(k)} = -\frac{iQ}{k - 2iQ} - \frac{iQk}{k^2 - 2iQk - P^2} \]
\[ \tilde{t}(k) = \frac{\dot{g}_2(k)}{\dot{f}(k)} = -\frac{iQ}{k - 2iQ} - \frac{iQ}{k^2 - 2iQk - P^2} = \frac{\dot{h}_2(k)}{\dot{f}(k)} = \tilde{t}'(k) \]
\[ r(k) = \frac{\dot{g}_1(k)}{\dot{f}(k)} = t(k) - 1. \]

(d) Hmmmm...let’s see. When \( m \to \infty \) we have \( Q \to 0 \) and \( P \to 0 \). So \( t(k) = \tilde{t}(k) = \tilde{t}'(k) = 0 \) and \( r(k) = -1 \). This makes perfect sense! There is no transmission – only reflection back into the top left channel, with an inverted \((r = -1)\) pulse. In this limit, the mass-spring system acts as an impenetrable wall. What about \( \kappa \to 0 \)? In this case \( P = 0 \) but \( Q \) is finite, and
\[ t(k) = -\frac{2iQ}{k - 2iQ}, \quad r(k) = -\frac{k}{k - 2iQ}, \quad \tilde{t}(k) = \tilde{t}'(k) = 0. \]

This corresponds to two decoupled strings – the spring is gone. Hence there is no transmission from string #1 to either side of string #2. There is transmission across the mass in string #1, though.

(e) In this limit, \( Q \to \infty \) and \( P \to \infty \), but \( P^2/Q = 2\kappa/\tau \) is finite. We define \( G \equiv \kappa/\tau \). We then have
\[ t(k) = \frac{k - iG/2}{k - iG}, \quad r(k) = \frac{iG/2}{k - iG}, \quad \tilde{t}(k) = \tilde{t}'(k) = -\frac{iG/2}{k - iG}. \]

Taking the modulus squared of each,
\[ T(k) = \frac{k^2 + G^2/4}{k^2 + G^2}, \quad R(k) = \tilde{T}(k) = \tilde{T}'(k) = \frac{G^2/4}{k^2 + G^2}. \]

Note that
\[ R(k) + T(k) + \tilde{T}(k) + \tilde{T}'(k) = 1, \]
which means that energy flux is conserved. Hooray!

(f) The Lagrangian density is
\[ \mathcal{L} = \frac{1}{2} \sigma(x) \dot{u}_1^2 + \frac{1}{2} \sigma(x) \dot{u}_2^2 - \frac{1}{2} \tau \dot{u}_1^2 - \frac{1}{2} \tau \dot{u}_2^2 - \frac{1}{2} \kappa (u_1 - u_2)^2 \delta(x), \]
where \( \sigma(x) = \sigma + m \delta(x) \). The equations of motion follow from the Euler-Lagrange equations,
\[ \frac{\partial \mathcal{L}}{\partial u_i} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{u}_i} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial u_i'} \right) = 0. \]
A particle of charge $e$ moves in the $(x, y)$ plane under the influence of a static uniform magnetic field $\mathbf{B} = B\hat{z}$. The potential is
$$U(r, \dot{r}) = e\phi(r) - \frac{e}{c} A(r) \cdot \dot{r}.$$ 
Choose the gauge
$$A = -\frac{1}{2}By\hat{x} + \frac{1}{2}Bx\hat{y}.$$ 

(a) Derive the Hamiltonian $H(x, y, p_x, p_y)$.

(b) Define the cyclotron coordinates $\left(\zeta_x, \zeta_y\right)$ and the guiding center coordinates $\{R_x, R_y\}$ as follows:
$$\zeta_x = \frac{1}{2}x - \frac{e}{ecB} p_y \quad R_x = \frac{1}{2}x + \frac{e}{ecB} p_y$$
$$\zeta_y = \frac{1}{2}y + \frac{e}{ecB} p_x \quad R_y = \frac{1}{2}y - \frac{e}{ecB} p_x.$$ 
Compute the Poisson brackets $\{\zeta_\mu, \zeta_\nu\}$, $\{R_\mu, R_\nu\}$, and $\{\zeta_\mu, R_\nu\}$, where $\mu, \nu = x$ or $y$.

(c) Show that $\pi_x$, the momentum conjugate to $\zeta_x$, is a constant times $\zeta_y$, and that $\kappa_y$, the momentum conjugate to $R_y$, is a constant times $R_x$.

(d) Write the equations of motion solely in terms of the cyclotron and guiding center coordinates. Note that
$$\phi(x, y) = \phi(R_x + \zeta_x, R_y + \zeta_y).$$

(e) When the cyclotron frequency $\omega_c = eB/mc$ is large, show that the motion of the cyclotron coordinates is approximately harmonic.

(f) Find the effective equations of motion for the slow guiding center coordinates by averaging over the fast oscillations of the cyclotron coordinates. Write down the effective potential and equations of motion for the pure guiding center dynamics.

(a) The Lagrangian, $L = T - U$, is
$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m (\dot{x}^2 + \dot{y}^2) - e\phi(x, y) + \frac{eB}{2c} (x\dot{y} - y\dot{x}).$$ 
The canonical momenta are given by
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{eB}{2c} y$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{eB}{2c} x.$$ 
The Hamiltonian is then
$$H(x, y, p_x, p_y) = p_x \dot{x} + p_y \dot{y} - L$$
$$= \frac{1}{2m} \left( p_x + \frac{eB}{2c} y \right)^2 + \frac{1}{2m} \left( p_y - \frac{eB}{2c} x \right)^2 + e\phi(x, y).$$
(b) Recall the definition of the Poisson bracket,

\[ \{F, G\} = \sum_\sigma \left( \frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right). \]

In our case, then

\[ \{F, G\} = \frac{\partial F}{\partial x} \frac{\partial G}{\partial p_x} - \frac{\partial F}{\partial p_x} \frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y} \frac{\partial G}{\partial y}. \]

We then find

\[ \{\zeta_x, \zeta_y\} = \frac{c}{eB}, \quad \{\zeta_x, R_y\} = -\frac{c}{eB} \]

and

\[ \{\zeta_x, R_x\} = \{\zeta_x, R_y\} = \{\zeta_y, R_x\} = \{\zeta_y, R_y\} = 0. \]

These may be compactly written as

\[ \{\zeta_\mu, \zeta_\nu\} = \frac{c}{eB} \varepsilon_{\mu\nu}, \quad \{R_\mu, R_\nu\} = -\frac{c}{eB} \varepsilon_{\mu\nu}, \quad \{\zeta_\mu, R_\nu\} = 0, \]

where \(\varepsilon_{\mu\nu} = i\sigma^y_{\mu\nu}\) is the rank-two antisymmetric tensor, with \(\varepsilon_{12} = -\varepsilon_{21} = 1\).

(c) From the Poisson bracket relations in part (c), we see that if we define

\[ \zeta_y = \frac{c}{eB} \pi_x, \quad R_x = \frac{c}{eB} \kappa_y, \]

then we have

\[ \{\zeta_x, \pi_x\} = \{\kappa_y, R_y\} = 1, \]

with all other (unrelated) brackets vanishing. This establishes the conjugacy of \((\zeta_x, \pi_x)\) and \((R_y, \kappa_y)\).

(d) Hamilton’s equations give

\[ \dot{\zeta}_x = \frac{\partial H}{\partial \pi_x} = \frac{c}{eB} \frac{\partial H}{\partial \zeta_y}, \quad \dot{\pi}_x = \frac{eB}{c} \zeta_y = -\frac{\partial H}{\partial \zeta_x}, \]

\[ \dot{\kappa}_y = \frac{eB}{c} \dot{R}_x = -\frac{\partial H}{\partial R_y}, \quad \dot{R}_y = \frac{\partial H}{\partial \kappa_y} = \frac{c}{eB} \frac{\partial H}{\partial R_x}. \]

hence Next, note that

\[ p_x + \frac{eB}{c} y = \frac{eB}{c} \zeta_y, \quad p_y - \frac{eB}{c} x = -\frac{eB}{c} \zeta_x, \]

so that

\[ H = \frac{1}{2} m \omega_c^2 (\zeta_x^2 + \zeta_y^2) + e \phi (R_x + \zeta_x, R_y + \zeta_y), \]
where \( \omega_c = eB/mc \) is the cyclotron frequency. We then have

\[
\dot{\zeta}_x = \omega_c \zeta_y - \frac{e}{m \omega_c} E_y (R_x + \zeta_x, R_y + \zeta_y)
\]

\[
\dot{\zeta}_y = -\omega_c \zeta_x + \frac{e}{m \omega_c} E_x (R_x + \zeta_x, R_y + \zeta_y)
\]

\[
\dot{R}_x = \frac{e}{m \omega_c} E_y (R_x + \zeta_x, R_y + \zeta_y)
\]

\[
\dot{R}_y = -\frac{e}{m \omega_c} E_x (R_x + \zeta_x, R_y + \zeta_y)
\]

where \( \mathbf{E}(r) = -\nabla \phi(r) \) is the electric field.

(e) In the large \( \omega_c \) limit, the equations for the cyclotron coordinates become

\[
\dot{\zeta}_x = \omega_c \zeta_y + \mathcal{O}(\omega_c^{-1}) \quad \dot{\zeta}_y = -\omega_c \zeta_x + \mathcal{O}(\omega_c^{-1})
\]

which, to lowest order, are instantly recognized as the equations for simple harmonic motion. Their solution is

\[
\zeta_x(t) = A \sin (\omega_c t + \delta) \quad \zeta_y(t) = A \cos (\omega_c t + \delta)
\]

(f) If the amplitude of the cyclotron oscillations is small on the scale over which \( \mathbf{E}(r) \) varies, we can expand the guiding center dynamics in a Taylor series in \( \zeta_x \) and \( \zeta_y \). This results in

\[
\dot{R}_x = \frac{e}{m \omega_c} \left\{ E_y (R_x, R_y) + \frac{\partial E_y}{\partial R_x} \zeta_x + \frac{\partial E_y}{\partial R_y} \zeta_y + \frac{1}{2} \frac{\partial^2 E_y}{\partial R_x^2} \zeta_x^2 + \frac{1}{2} \frac{\partial^2 E_y}{\partial R_y^2} \zeta_y^2 + \ldots \right\}
\]

\[
\dot{R}_y = -\frac{e}{m \omega_c} \left\{ E_x (R_x, R_y) + \frac{\partial E_x}{\partial R_x} \zeta_x + \frac{\partial E_x}{\partial R_y} \zeta_y + \frac{1}{2} \frac{\partial^2 E_x}{\partial R_x^2} \zeta_x^2 + \frac{1}{2} \frac{\partial^2 E_x}{\partial R_y^2} \zeta_y^2 + \ldots \right\}
\]

Averaging over the fast cyclotron degrees of freedom over the period \( T = 2\pi/\omega_c \), we have

\[
\langle \zeta_x^2(t) \rangle = \langle \zeta_y^2(t) \rangle = \frac{1}{2} A^2, \quad \langle \zeta_x(t) \zeta_y(t) \rangle = 0
\]

Thus, the dominant, ‘slow part’ of the guiding center dynamics, given by \( \mathcal{R}_{x,y} = \langle R_{x,y} \rangle \), obeys

\[
\dot{R}_x = -\frac{c}{B} \frac{\partial \phi}{\partial R_y}, \quad \dot{R}_y = +\frac{c}{B} \frac{\partial \phi}{\partial R_x}
\]
with
\[
\tilde{\phi} = \phi + \frac{1}{4} \langle \zeta^2 \rangle \nabla^2 \phi
\]
\[
= \phi + \frac{1}{4} A^2 \nabla^2 \phi .
\]
That is, the motion of the guiding center is in an effective electrical potential which includes information about the local neighborhood of the coordinates \((x, y)\), obtained via averaging over the excursions of the fast cyclotron variables.

[4] A particle of mass \(m\) moves in the potential \(U(q) = A \, |q|\). The Hamiltonian is thus
\[
H_0(q, p) = \frac{p^2}{2m} + A |q| ,
\]
where \(A\) is a constant.

(a) List all independent conserved quantities.

(b) Show that the action variable \(J\) is related to the energy \(E\) according to \(J = \beta E^{3/2}/A\), where \(\beta\) is a constant, involving \(m\). Find \(\beta\).

(c) Find \(q = q(\phi, J)\) in terms of the action-angle variables.

(d) Find \(H_0(J)\) and the oscillation frequency \(\nu_0(J)\).

(e) The system is now perturbed by a quadratic potential, so that
\[
H(q, p) = \frac{p^2}{2m} + A |q| + \epsilon B q^2 ,
\]
where \(\epsilon\) is a small dimensionless parameter. Compute the shift \(\Delta \nu\) to lowest nontrivial order in \(\epsilon\), in terms of \(\nu_0\) and constants.

\textbf{Solution :}

(a) The only conserved quantity is the Hamiltonian itself:
\[
\frac{dH_0}{dt} = \frac{\partial H_0}{\partial t} = 0 .
\]
We write \(H_0(q, p) = E\), the total energy. Clearly \(E \geq 0\), and \(E = 0\) is particularly boring.

(b) Since the energy is conserved, we have
\[
p(q) = \pm \sqrt{2m(E - A|q|)} .
\]
There are two turning points, at $q_{\pm}(E) = E/A$. We can integrate to get the action:

$$J = \frac{1}{2\pi} \oint_C p \, dq$$

$$= \frac{2}{\pi} \int_0^{E/A} dq \sqrt{2m(E - Aq)}$$

$$= \frac{4\sqrt{2m}}{3\pi A} E^{3/2} \equiv \frac{\beta}{A} E^{3/2},$$

with $\beta = 4\sqrt{2m}/3\pi$. Note that the integral over a complete cycle is written above as four times the integral over a quarter cycle, i.e. from $q = 0$ to $q = q_+(E) = E/A$.

(c) We first obtain the characteristic function $W(q, E(J))$. We have

$$p = \frac{dW}{dq} = \pm \sqrt{2m(E - A|q|)} \Rightarrow W(q) = \pm \frac{\pi \beta}{2A} (E - A|q|)^{3/2} \text{sgn}(q),$$

where we’ve used $\frac{2\sqrt{2m}}{3\pi} = \frac{\pi}{2}\beta$. The angle variable is

$$\phi = \frac{\partial W}{\partial J} = \frac{\partial W}{\partial E} \frac{\partial E}{\partial J} = \mp \frac{\pi \beta^{1/3}}{2A^{1/3}} J^{-1/3} (E - A|q|)^{1/2} \text{sgn}(q).$$

Squaring, we find

$$\left(\frac{2\phi}{\pi}\right)^2 = \left(\frac{\beta}{AJ}\right)^{2/3} (E - A|q|)$$

$$= 1 - \left(\frac{\beta^2 A}{J^2}\right) |q|.$$

Thus,

$$q(\phi, J) = \frac{J^{2/3}}{\left(\beta^2 A\right)^{1/3}} \left\{1 - \frac{4}{\pi^2} \phi^2 \right\} \quad \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

This is valid on the interval $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where $q$ is positive. In fact, this is all we need to solve the problem, but it is worthwhile writing down the continuation of this relation for the other half of the cycle, i.e. for $\phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. This can be done by inspection, taking advantage of the symmetry of the orbit $C$:

$$q(\phi, J) = \frac{J^{2/3}}{\left(\beta^2 A\right)^{1/3}} \left\{\frac{4}{\pi^2} (\phi - \pi)^2 - 1 \right\} \quad \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

(d) We have

$$H_0(J) = E = \beta^{-2/3} A^{2/3} J^{2/3},$$

so

$$\nu_0(J) = \frac{\partial H_0(J)}{\partial J} = \frac{2}{3} \beta^{-2/3} A^{2/3} J^{-1/3}.$$
(e) Expressed in terms of the action-angle variables \((\phi, J)\), the perturbing Hamiltonian is 
\[ H_1(\phi, J) = B q^2 = B \cdot \left( \frac{J^2}{\beta^2 A} \right)^{2/3} \left( 1 - \frac{4}{\pi} \phi^2 \right)^2. \]

This holds for all \(\phi\) provided we periodically extend the function \(\phi^2\) from the interval \(\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) to the entire real line. Due to the parity \((q \rightarrow -q)\) symmetry, we can average over a quarter cycle, and we obtain
\[ \langle H_1(\phi, J) \rangle = B \cdot \left( \frac{J^2}{\beta^2 A} \right)^{2/3} \int_0^1 ds (1 - s^2)^2 = \frac{8B}{15(\beta^2 A)^{2/3}} J^{4/3}, \]
where we’ve substituted \(s = \frac{2}{\pi} \phi\). The energy shift is \(\Delta E = \epsilon \langle H_1 \rangle\). Thus,
\[
\nu(J) = \nu_0(J) + \frac{32}{45} \epsilon \frac{BJ^{1/3}}{(\beta^2 A)^{3/2}} \\
= \nu_0(J) + \epsilon \cdot \frac{2\pi^2 B}{15 m} \cdot \frac{1}{\nu_0(J)}.
\]

[5] Provide short but accurate answers to the following questions:

(a) Write down a generating function for a canonical transformation which generates a dilation: \(Q = \lambda q, P = \lambda^{-1} p\).

(b) Give an explicit example of a two-dimensional phase flow which is invertible but not volume preserving.

(c) What is Noether’s theorem? Give an example and be explicit.

(d) Consider the Lagrangian,
\[ L = \frac{1}{2} m_\perp(t) (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_z \dot{z}^2 - \frac{1}{4} (x^4 + 2x^2 y^2 + y^4), \]
where \(m_\perp(t)\) is time-dependent. List and provide expressions for all conserved quantities.

(e) How are the Euler angles defined?

(f) Explain the content and physics of the ‘tennis racket theorem’.

(a) \(F_2(q, P) = \lambda q P\) does the trick:
\[
p = \frac{\partial F_2}{\partial q} = \lambda P, \quad Q = \frac{\partial F_2}{\partial P} = \lambda q.
\]
Thus, $Q = \lambda q$ and $P = p/\lambda$. This is not unique.

(b) Any dissipative flow will do. Consider, for example, a damped linear oscillator:

$$\ddot{x} + 2b \dot{x} + \omega_0^2 x = 0 \Rightarrow \frac{d}{dt} \left( \dot{x} \right) = \left( -\omega_0^2 x - 2b \dot{x} \right).$$

In this linear case we can explicitly compute the rate at which phase space volumes collapse. Writing the above $N = 2$ system as $\dot{\varphi} = \mathbf{X}(\varphi)$, we have $\nabla \cdot \mathbf{X} = -2b$, which means $\Omega(t) = \Omega(0) \exp(-2bt)$, where $\Omega(t)$ is the volume of a region of phase space, each point in which evolves according to the above dynamics.

(c) Noether’s theorem states that in every organizational hierarchy, each individual is eventually promoted to a level of personal incompetence. Or maybe that’s the ‘Peter Principle’. Try again.

(c) Noether’s theorem, as applied to particle mechanics, says to each and every continuous symmetry of a mechanical system is associated a conserved quantity. Let $\tilde{q}(q, \lambda)$ be a one-parameter family of transformations of the generalized coordinates $q = \{q_1, \ldots, q_n\}$, parametrized by $\lambda$, with $\tilde{q}_\sigma(q, \lambda = 0) = q_\sigma$, i.e. $\lambda = 0$ is the identity transformation. Then we must have

$$0 = \frac{d}{d\lambda} \left. L(\tilde{q}, \dot{\tilde{q}}, t) \right|_{\lambda=0} = \frac{\partial L}{\partial q_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \lambda} \bigg|_{\lambda=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \lambda} \bigg|_{\lambda=0}$$

$$= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_\sigma} \right) \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \lambda} \bigg|_{\lambda=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{d}{dt} \left( \frac{\partial \dot{q}_\sigma}{\partial \lambda} \right) \bigg|_{\lambda=0}$$

$$= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_\sigma} \right) \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \lambda} \bigg|_{\lambda=0} .$$

Thus, there is an associated conserved charge

$$Q = \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \lambda} \bigg|_{\lambda=0} .$$

As an example consider the very next problem, in part (d) ...

(d) Clearly $z$ is cyclic (does not appear) in $L$, so its conjugate momentum $p_z = m_z \dot{z}$ is conserved. Since $L$ is explicitly time-dependent, the Hamiltonian, which is the total energy for this system, is not conserved. However, astute readers will notice that

$$V(x, y) = \frac{1}{4} (x^4 + 2x^2y^2 + y^4) = \frac{1}{4} (x^2 + y^2)^2 ,$$

is invariant under a rotation,

$$x \rightarrow \tilde{x}(\lambda) = x \cos \lambda + y \sin \lambda$$
$$y \rightarrow \tilde{y}(\lambda) = -x \sin \lambda + y \cos \lambda .$$
The associated conserved charge, from Noether’s theorem, is
\[
Q = \left. \frac{\partial L}{\partial \dot{x}} \frac{\partial}{\partial x} \right|_{\lambda=0} + \left. \frac{\partial L}{\partial \dot{y}} \frac{\partial}{\partial y} \right|_{\lambda=0} = m_1(t) \dot{x} y - m_1(t) \dot{y} x.
\]

One sees that \(Q = -L_z\), where \(L_z\) is the angular momentum. So, only \(L_z = m_1(t) \dot{x} y - m_1(t) \dot{y} x\) and \(p_z = m_z \dot{z}\) are conserved.

\(e)\) The Euler angles are a set of generalized coordinates appropriate for describing the orientation of a rigid body. Starting with some fiducial orientation, we first rotate about the \(\hat{z}\) axis by an angle \(\phi\). Next, we rotate about the new \(\hat{x}\) axis by an angle \(\theta\). Finally, we rotate about the new new \(\hat{z}\) axis by an angle \(\psi\). The result is:

\[
R(\phi, \theta, \psi) = R(\hat{e}^0_3, \phi) R(\hat{e}^j_1, \theta) R(\hat{e}^0_3, \psi)
\]

\[
= \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi & \cos \psi \sin \phi + \sin \psi \cos \theta \cos \phi & \sin \psi \sin \theta \\
-\sin \psi \cos \phi - \cos \psi \cos \theta \sin \phi & \cos \psi \sin \phi + \cos \psi \cos \theta \cos \phi & \cos \psi \sin \theta \\
\sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta
\end{pmatrix}.
\]

\(f)\) The content of the ‘tennis racket theorem’ is this: an asymmetric top with three distinct principal moments of inertia \(I_1 < I_2 < I_3\) moving under the influence of no net external torque is rotationally marginally stable when rotating about the body fixed axes \(\hat{e}_1\) and \(\hat{e}_3\), but is unstable when rotating about \(\hat{e}_2\). This is easily shown by solving Euler’s equations,

\[
\begin{align*}
I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 + N_1^{\text{ext}} \\
I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_3 \omega_1 + N_2^{\text{ext}} \\
I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 + N_3^{\text{ext}},
\end{align*}
\]

setting \(N^{\text{ext}} = 0\) and perturbing about an initial solution \(\omega = \omega_0 \hat{e}_j + \delta \omega\). To lowest order in \(\delta \omega\), one finds the component \(\omega_j\) is constant, while the two other components oscillate with a frequency \(\Omega\), where

\[
\Omega^2 = \frac{(I_j - I_k)(I_j - I_l)}{I_k I_l},
\]

where \(k\) and \(l\) are the indices for the other two principal axes. If \(I_j\) is the middle of the three principal moments of inertia, then \(\Omega^2 < 0\), corresponding to exponential deviation from the initial conditions, \textit{i.e.} instability. If \(\Omega^2 > 0\), the oscillations are stable. (The perturbation is called marginally stable because it does not diverge away from the initial condition, but neither does it relax to the initial condition.)