[1] A point particle of mass $m$ in two-dimensions moves along a one-dimensional surface under the influence of gravity $g = -g \hat{y}$. The equation of the surface is

$$y = x - \frac{x^3}{3a^2}.$$ 

The particle is released from rest at a point along the curve $(x_0, y_0)$. The particle flies off the curve at $x = a$. Determine $y_0$.

[2] Consider the two coupled strings of fig. 1. Both strings are described by identical mass density $\sigma$ and tension $\tau$. On each string, at $x = 0$, a point mass $m$ is affixed. The two masses are connected via a spring of constant $\kappa$. When the two masses are identically displaced, i.e. when $u_1(0, t) = u_2(0, t)$, the spring is unstretched. There are no other forces aside from the tension in the strings and the restoring force of the spring.

Figure 1: Two identical strings with masses $m$ at $x = 0$, coupled via a spring of constant $\kappa$.

(a) Let $u_i(x, t)$ be the displacement field of each string ($i = 1, 2$). Write down the equations of motion for the two masses.

(b) Taking advantage of the symmetry under interchange of the two strings, define sum and difference fields,

$$u_\pm(x, t) \equiv u_1(x, t) \pm u_2(x, t),$$

and rewrite the equations from part (a) in terms of $u_\pm(0, t)$.

(c) In the distant past, a pulse of shape $f(\xi)$ is incident from the left on string #1. Given the definition of the functions $g_i(\xi)$ and $h_i(\xi)$ implicit in the figure, find the complex reflection and transmission coefficients,

$$r(k) = \frac{\hat{g}_1(k)}{f(k)}, \quad t(k) = \frac{\hat{h}_1(k)}{f(k)}, \quad \tilde{t}(k) = \frac{\hat{g}_2(k)}{f(k)}, \quad \tilde{t}'(k) = \frac{\hat{h}_2(k)}{f(k)}.$$
where \( \hat{f}(k) \) is the Fourier transform of \( f(\xi) \), etc.\(^1\) For notational convenience, you should define \( Q = \tau/mc^2 \) and \( P = \sqrt{2\kappa/mc^2} \).

(d) Do your answers make sense in the limits \( m \to \infty \) and \( \kappa \to 0 \) ?

(e) Consider the limit \( m \to 0 \) with \( \kappa, \tau, \) and \( \sigma \) held fixed. Find the transmission and reflection coefficients \( T = |t|^2, R = |r|^2, \tilde{T} = |\tilde{t}|^2, \tilde{T}' = |\tilde{t}'|^2 \), and show that energy flux is conserved.

(f) Write down the Lagrangian density \( \mathcal{L} \) for this system.

[3] A particle of charge \( e \) moves in the \((x, y)\) plane under the influence of a static uniform magnetic field \( B = B\hat{z} \). The potential is

\[
U(r, \dot{r}) = e \phi(r) - \frac{e}{c} A(r) \cdot \dot{r} .
\]

Choose the gauge

\[
A = -\frac{1}{2}By \, \hat{x} + \frac{1}{2}Bx \, \hat{y} .
\]

(a) Derive the Hamiltonian \( H(x, y, p_x, p_y) \).

(b) Define the cyclotron coordinates \((\zeta_x, \zeta_y)\) and the guiding center coordinates \(\{R_x, R_y\}\) as follows:

\[
\zeta_x = \frac{1}{2} x - \frac{e}{c} B p_y \quad \quad \quad \quad R_x = \frac{1}{2} x + \frac{e}{c} B p_y
\]

\[
\zeta_y = \frac{1}{2} y + \frac{e}{c} B p_x \quad \quad \quad \quad R_y = \frac{1}{2} y - \frac{e}{c} B p_x .
\]

Compute the Poisson brackets \( \{\zeta_\mu, \zeta_\nu\}, \{R_\mu, R_\nu\}, \) and \( \{\zeta_\mu, R_\nu\} \), for all possible pairings of \( \mu, \nu = x \) or \( y \).

(c) Show that \( \pi_x \), the momentum conjugate to \( \zeta_x \), is a constant times \( \zeta_y \), and that \( \kappa_y \), the momentum conjugate to \( R_y \), is a constant times \( R_x \).

(d) Write the equations of motion solely in terms of the cyclotron and guiding center coordinates. Note that

\[
\phi(x, y) = \phi(R_x + \zeta_x, R_y + \zeta_y) .
\]

(e) When the cyclotron frequency \( \omega_c = eB/mc \) is large, show that the motion of the cyclotron coordinates is approximately harmonic.

\(^1\)Even though the \( g_2 \) wave moves to the left, we consider this transmission from one branch of string to another, rather than reflection into the same branch.
[4] A particle of mass $m$ moves in the potential $U(q) = A |q|$. The Hamiltonian is thus

$$H_0(q,p) = \frac{p^2}{2m} + A |q|,$$

where $A$ is a constant.

(a) List all independent conserved quantities.

(b) Show that the action variable $J$ is related to the energy $E$ according to $J = \beta E^{3/2}/A$, where $\beta$ is a constant, involving $m$. Find $\beta$.

(c) Find $q = q(\phi, J)$ in terms of the action-angle variables.

(d) Find $H_0(J)$ and the oscillation frequency $\nu_0(J)$.

(e) The system is now perturbed by a quadratic potential, so that

$$H(q,p) = \frac{p^2}{2m} + A |q| + \epsilon B q^2,$$

where $\epsilon$ is a small dimensionless parameter. Compute the shift $\Delta \nu$ to lowest nontrivial order in $\epsilon$, in terms of $\nu_0$ and constants.

[5] Provide short but accurate answers to the following questions:

(a) Write down a generating function for a canonical transformation which generates a dilation: $Q = \lambda q$, $P = \lambda^{-1} p$.

(b) Give an explicit example of a two-dimensional phase flow which is invertible but not volume preserving.

(c) What is Noether’s theorem? Give an example and be explicit.

(d) Consider the Lagrangian,

$$L = \frac{1}{2} m_{\perp}(t) (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_{\perp} \dot{z}^2 - \frac{1}{4} (x^4 + 2 x^2 y^2 + y^4),$$

where $m_{\perp}(t)$ is time-dependent. List and provide expressions for all conserved quantities.

(e) How are the Euler angles defined?

(f) Explain the content and physics of the ‘tennis racket theorem’. 