[1] Consider a planar pendulum consisting of a point mass $m$ at the end of a massless rigid rod of length $\ell$. Treat the problem using 2D polar coordinates $(r, \phi)$ along with the constraint $r = \ell$.

(a) What are the equations of motion?

(b) Identify all conserved quantities.

(c) Suppose the pendulum is released from rest from an angle $\phi_0$. Find the force of constraint, i.e. the tension in the rod, as a function of the angular position $\phi$.

**SOLUTION** :

(a) The Lagrangian is

$$ L = \frac{1}{2} m (\ddot{r}^2 + r^2 \dot{\phi}^2) + mgr \cos \phi $$

and the constraint is $G(r, \phi) = r - \ell = 0$. The equations of motion are

$$ m \ddot{r} = mr \dot{\phi}^2 + mg \cos \phi + \lambda $$

$$ mr^2 \ddot{\phi} + 2mr \dot{r} \dot{\phi} = -mgr \sin \phi. $$

(b) The only conserved quantity is the total energy $E = T + U$. Implementing the constraint, this says

$$ E = \frac{1}{2} m \dot{\phi}^2 - m g \ell \cos \phi. $$

With initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$, we have $E = -mg \ell \cos \phi_0$.

(c) The tension is $T = -\lambda$, hence

$$ T = m \ell \dot{\phi}^2 + mg \cos \phi $$

$$ = (3 \cos \phi - 2 \cos \phi_0) mg. $$
A mass $m$ moves frictionlessly under the influence of gravity along the curve $y = x^2/2a$. Attached to the mass is a massless rigid rod of length $\ell$, at the end of which is an identical mass $m$. The rod is constrained to swing in the $(x, y)$ plane, as depicted in the figure below.

(a) Choose as generalized coordinates $x$ and $\phi$. Find the kinetic energy $T$ and potential energy $U$.

(b) For small oscillations, find the $T$ and $V$ matrices. It may be convenient to define $\Omega_1 \equiv \sqrt{g/a}$ and $\Omega_2 \equiv \sqrt{g/\ell}$.

(c) Find the eigenfrequencies of the normal modes of oscillation.

(d) Suppose $\Omega_1 = \sqrt{3}\Omega_0$ and $\Omega_2 = 2\Omega_0$, where $\Omega_0$ has dimensions of frequency. Find the modal matrix.

**SOLUTION:**

(a) The coordinates of the mass on the curve are $(x_1, y_1) = (x, x^2/2a)$. Note $\dot{y} = (x/a) \dot{x}$. The coordinates for the hanging mass are $(x_2, y_2) = (x + \ell \sin \phi, x^2/2a - \ell \cos \phi)$. The kinetic energy is

$$
T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\
= m \left( 1 + \frac{a^2}{x^2} \right) x^2 + \frac{1}{2} m \ell^2 \dot{\phi}^2 + m \ell \left( \cos \phi + \frac{x}{a} \sin \phi \right) \dot{x} \dot{\phi} 
$$

The potential energy is

$$
U = mg(y_1 + y_2) = \frac{mg}{a} x^2 - mgl \cos \phi 
$$

(b) Equilibrium occurs for $x = \phi = 0$, hence

$$
T_{\sigma\sigma'} = \left. \frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}} \right|_{\dot{q}} = \begin{pmatrix} 2m & m\ell \\ m\ell & m\ell^2 \end{pmatrix}
$$
and

\[ V_{\sigma\sigma'} = \left. \frac{\partial^2 T}{\partial q_\sigma \partial q_{\sigma'}} \right|_q = \begin{pmatrix} 2m\Omega^2_1 & 0 \\ 0 & ml^2\Omega^2_2 \end{pmatrix} . \]

(c) We set \( P(\omega^2) = \det(\omega^2 T - V) = 0 \), with

\[ \omega^2 T - V = \begin{pmatrix} 2m(\omega^2 - \Omega^2_1) \\ ml\omega^2 \\ ml(\omega^2 - \Omega^2_2) \end{pmatrix} \]

Thus,

\[ P(\omega^2) = m^2 l^2 \left\{ \omega^4 - 2(\Omega^2_1 + \Omega^2_2)\omega^2 + 2\Omega^2_1\Omega^2_2 \right\} . \]

Solving the quadratic equation, we have the two normal mode frequencies

\[ \omega^2_{\pm} = \Omega^2_1 + \Omega^2_2 \pm \sqrt{\Omega^4_1 + \Omega^4_2} . \]

(d) With \( \Omega_1 = \sqrt{3} \Omega_0 \) and \( \Omega_2 = 2\Omega_0 \), we have \( \omega^2_+ = 12\Omega^2_0 \) and \( \omega^2_- = 2\Omega^2_0 \). We then solve for the eigenvectors using \( (\omega^2_i T - V)_{\sigma\sigma'} A_{\sigma' i} = 0 \). From the form of \( \omega^2 T - V \), we see that

\[ A_{2,i} = \frac{\ell^{-1}\omega^2_i}{\Omega^2_2 - \omega^2_i} A_{1,i} \]

and imposing the normalization \( A^T A = I \), we have

\[ A = \frac{1}{\sqrt{5m}} \begin{pmatrix} -3\ell^{-1} & 1 \\ 2 & \ell^{-1} \end{pmatrix} . \]