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Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 7 – Solutions

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to discuss scattering problems and the Born approximation.

Problems

The basic set-up of scattering is to suppose you have some incoming state $|\psi_0\rangle$ which a plane-wave/free particle which then interacts with a potential V to produce a scattered state $|\psi_s\rangle$. You want construct the solution to the full Hamiltonian so that in the limit of $V \to 0$ you recover a state with the same energy.

To start you have $(E - H_0)|\psi_0\rangle = 0$ where $H_0 = \frac{k^2}{2m}$ and $\langle x|\psi_0\rangle = \frac{1}{\sqrt{2\pi}} e^{\mathbf{i}k\cdot x}$

We must solve $(E - H_0 - V)|\psi\rangle = 0$ where for each energy E there's a different incoming and outgoing state. Define $|\psi_s\rangle = |\psi\rangle - |\psi_0\rangle$ and plug in: $(E - H_0)|\psi_s\rangle = V|\psi\rangle$

One can 'solve' for $|\psi_s\rangle$ by defining the formal inverse $(E - H_0)^{-1}$ and then construct the full solution as:

$$|\psi\rangle = |\psi_s\rangle + |\psi_0\rangle = \frac{V}{(E - H_0)}|\psi\rangle + |\psi_0\rangle \tag{1}$$

This formal inverse is called a 'Green's function'. You'll note however that this expression is singular when the eigenvalue of H_0 is E so we can redefine things with an infinitesimal correction:

$$G_0(E) = \lim_{\epsilon \to 0} (E - H_0 + \mathbf{i}\epsilon)^{-1} \tag{2}$$

Just as in our discussion of time-dependent perturbation theory, you can recursively substitute expression 1 into itself to generate an expansion in V. This is the Born series:

$$|\psi\rangle = (\mathbb{1} + G_0 V + G_0 V G_0 V + \cdots)|\psi_0\rangle \tag{3}$$

OK! So now you should ask, how can I actually calculate G_0? Answer: Multiply by $\mathbb{1}$

$$G_0 = G_0 \mathbb{1} = \sum_{E'} G_0 |E'\rangle\langle E'| = \sum_{E'} \frac{|E'\rangle\langle E'|}{E - E' + \mathbf{i}\epsilon} \tag{4}$$

The sum is schematic and we're sliding over difficulties like the continuum, degeneracy, and boundstates but it is correct. We've reduced the problem to some sum/integral.

1. **Simplest Case**

   Consider a one dimensional particle incident on a potential $V(x) = V_0 \delta(x - x_0)$

   (a) Construct $G_0(x, x') \equiv \langle x|G_0|x'\rangle$ for a free particle of $E = \frac{k^2}{2m}$ using 4

   Directly substituting: $G_0(x, x') = \int \mathrm{d}E' \frac{\langle x|E'\rangle\langle E'|x'\rangle}{E-E'+\mathbf{i}\epsilon} = \frac{1}{2\pi} \int \mathrm{d}k' \frac{e^{\mathbf{i}k'(x-x')}}{E - \frac{k'^2}{2m} - \mathbf{i}\epsilon}$

   $= -\frac{m}{\pi} \int \mathrm{d}k' \frac{e^{\mathbf{i}k'(x-x')}}{k'^2 - k^2 + \mathbf{i}\epsilon} = -\frac{m}{\pi} \int \mathrm{d}k' \frac{e^{\mathbf{i}k'(x-x')}}{(k'+k+\mathbf{i}\epsilon)(k'-k-\mathbf{i}\epsilon)}$

   To do this integral we should switch to complex $k' = k_R + \mathbf{i}k_I$ where the numerator will now have a term $e^{-k_I(x-x')}$ decaying at $\pm\mathbf{i}\infty$ but what side of the complex plane we need depends on if $x' > x$ or $x > x'$

   This subtlety aside we note the pole appears at $k' = \pm(k + \mathbf{i}\epsilon)$ and supposing $x > x'$ we can pick everything to be positive.

   By Cauchy's theorem for simple poles $\oint \mathrm{d}z \frac{f(z)}{h(z)} = 2\pi\mathbf{i}\frac{f(p)}{h'(p)}$ we have

   $G_0(x, x') = -\mathbf{i}m\frac{e^{\mathbf{i}k(x-x')}}{k}$

   For this choice of potential we needn't resort to perturbation theory.

   (b) Using 1 write a form of $\psi(x) = \langle x|\psi\rangle$ in terms of $G_0(x, x_0)$

   Hint: Write $\psi(x_0)$ in terms of $G_0(x_0, x_0)$

   The LS formula gives $\psi(x) = \psi_0(x) + \int \mathrm{d}x' G_0(x, x')V(x')\psi(x)$ which is simply $= \psi_0(x) + V_0 G_0(x, x_0)\psi(x_0)$ where we can find $\psi(x_0)$ by plugging in $x = x_0$

   $\psi(x_0) = \psi_0(x_0) + V_0 G_0(x_0, x_0)\psi(x_0) \implies \psi(x_0) = \frac{\psi_0(x_0)}{1 - V_0 G_0(x_0, x_0)}$

   (c) Using the form of G_0 you derived in part (a) express $\psi(x)$ directly in terms of k

   $\psi(x) = \psi_0(x) + \frac{V_0\psi_0(x_0)}{1 - V_0 G_0(x_0, x_0)} G_0(x, x_0) = e^{\mathbf{i}kx} + \frac{V_0 e^{\mathbf{i}kx_0}}{1 + \mathbf{i}V_0\frac{m}{k}}(-\mathbf{i}\frac{m}{k}e^{\mathbf{i}k(x-x_0)}) = e^{\mathbf{i}kx} + \frac{e^{\mathbf{i}kx}}{\mathbf{i}\frac{k}{mV_0} - 1}$

   (d) Solve for the transmission probability $|\psi(x \to \infty)|^2$

   $\psi(x) = \frac{\mathbf{i}\frac{k}{mV_0}}{1 - \mathbf{i}\frac{k}{mV_0}} e^{\mathbf{i}kx}$ so the transmission probability is just $|\frac{\mathbf{i}\frac{k}{mV_0}}{1 - \mathbf{i}\frac{k}{mV_0}}|^2 = \frac{(ka)^2}{1 + (ka)^2}$ for $a \equiv \frac{1}{mV_0}$