University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 3 – Solutions

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to learn about spin, rotations, representations, and all that jazz.

Problems

Suppose we are studying a system with a rotational symmetry. So we need understand how to *represent* this symmetry on our Hilbert space of states. This involves creating matrices which do all the things we expect.

1. **Do a Barrel Roll**

   Recall that in 3-dimensional space[1] we can derive the following rotation matrices from geometry:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

   where R_i is a rotation about the i-th axis by an angle θ. Consider a rotation with an infinitesimal $\theta = \delta\theta$.

   (a) Express each rotation in 1 as $R_i(\theta = \delta\theta) = \mathbb{1} - \mathbf{i}(\delta\theta)X_i$ for some matrices X_i. These are the *generators* of rotations as we'll see in a moment.[2]

   Use small angle approximation $\cos\theta \approx 1$ and $\sin\theta \approx \theta$

[1] Euclidean. Over $\mathbb{R}$. Don't get cheeky.

[2] Note that the factor of $\mathbf{i}$ is conventional.

Spoilers. The form of X_i is simply:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} \\ 0 & \mathbf{i} & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & \mathbf{i} \\ 0 & 0 & 0 \\ -\mathbf{i} & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & -\mathbf{i} & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

(b) Show explicitly that each X_i is Hermitian: $X^\dagger = X$

Obvious

(c) I claim that the X_i of 2 satisfy the following algebra[3]

$$[X_i, X_j] = \mathbf{i}\epsilon^{ijk} X_k \tag{3}$$

Convince yourself of this by checking a few examples.

$X_1 X_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $X_2 X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This implies $[X_1, X_2] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{i} X_3$ and so on.

Given a hermitian matrix X one can construct a unitary matrix $U = e^{-\mathbf{i}Xa}$ which 'evolves' a state by an amount a. For example the Hamiltonian $\hat{H}$ is hermitian and leads to the 'time-evolution' operator $U = e^{-\mathbf{i}\hat{H}t}$.

In this way $\hat{H}$ *generates* time evolution. Can you guess where this is going?

(d) Consider the unitary matrices given by $U_i = e^{-\mathbf{i}X_i\theta}$ for each X_i in 2. Show, using Taylor's theorem, that $U_i = R_i$; they are the rotation matrices of 1.

Let's do this for X_1, the rest follow very similarly.

$U_1 = e^{-\mathbf{i}X_1\theta} = e^{\theta A}$ for $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Now let's Taylor expand:

$e^{\theta A} = \sum_{n=0}^{\infty} \frac{\theta^n A^n}{n!} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{\theta^n A^n}{n!}$ where now we need to note the following facts:

$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv -B \qquad A^3 = -A \qquad A^4 = B \qquad A^5 = A$ and so on.

This allows us to split the infinite sum into evens and odds and then pull out our A and B matrices.

$e^{\theta A} = \mathbb{1} + \sum_{n,even} \frac{\theta^n A^n}{n!} + \sum_{n,odd} \frac{\theta^n A^n}{n!} = \mathbb{1} + \sum_{n,even} \frac{(-1)^{\frac{n}{2}}\theta^n}{n!} B + \sum_{n,odd} \frac{(-1)^{\frac{n-1}{2}}\theta^n}{n!} A$
$= \mathbb{1} + (\cos\theta - 1)B + \sin\theta A = R_x$

2. **What *is* Spin?**

The fact there are spin-$\frac{1}{2}$ particles is one of the most deeply quantum features of nature.

[3]This is known as a Lie algebra

We can think of the spin of an electron as an additional degree of freedom. This is represented quantum mechanically is a two dimensional Hilbert space $\mathcal{H}_2$ spanned by two vectors $\{|\uparrow\rangle, |\downarrow\rangle\}$

Now, how can we represent rotations on this space?

Consider the following matrices:

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

These, up to that factor of $\frac{1}{2}$, are known as the Pauli matrices.

(a) Show that S_i are hermitian. Show explicitly that the following algebra is satisfied:

$$[S_i, S_j] = \mathbf{i}\epsilon^{ijk} S_k \tag{5}$$

This is the same algebra as 3, between the generators of rotations![4] Together these imply we are constructing something like angular momentum.

Just do it.

Now let's construct the analog of rotation matrices for these objects.

(b) Define $U_i = e^{-\mathbf{i}\theta S_i}$ and write a simple matrix expression for it.

Hint: Use the fact $\sigma_i^2 = \mathbb{1}$ where σ_i is a Pauli matrix.

$e^{-\mathbf{i}\theta S_i} = e^{-\mathbf{i}\frac{\theta}{2}\sigma_i} = \mathbb{1}\cos\frac{\theta}{2} - \mathbf{i}\sigma_i \sin\frac{\theta}{2}$

(c) Now consider $U_i(\theta = 2\pi)$, what has happened?

$U_i(\theta = 2\pi) = \mathbb{1}\cos\pi - \mathbf{i}\sigma_i \sin\pi = -\mathbb{1}$

We have gone around a complete rotation and picked up a minus sign!

[4] Fancy math point, this is the statement $SO(3)$ and $SU(2)$ have the same Lie algebra.