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Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 2 – Solutions

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to remind ourselves about Hydrogen. For this worksheet I'll do my best to keep factors of $\hbar$.

Problems

1. **Quantum Gravity?**

   Let's do something silly. Consider the Earth-Sun as a hydrogen atom like system.

   (a) What potential should enter the Hamiltonian? What replacement does one need to make compared to the hydrogen atom?
   $U(r) = -G\frac{Mm}{r}$ so replace $\frac{e^2}{4\pi\epsilon_0} \to GMm$ where M is the Sun's mass and m is the Earth's mass.

   (b) What is the quantity analogous to the 'Bohr radius' a_g for this system? What is it numerically?
   Recall the Hydrogen Bohr radius is $a_0 \equiv \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{\mu}$ where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass. Using the above substitution $a_g = \frac{\hbar^2}{GMm\mu}$ using Sun/Earth masses in μ
   Chugging in numbers $a_g \approx 10^{-138}[m]$

   (c) Write down the expression for the energy spectrum E_n. Equate this quantity to the classical result for energy in simple circular motion to show $n^2 = \frac{r_0}{a_g}$. What is an estimate for n at r_0 the current orbiting radius?
   $E_n = -\frac{\mu}{2\hbar^2}(GMm)^2\frac{1}{n^2}$ doing the same as before. Recall the total energy classically is $E = \frac{1}{2}mv^2 - \frac{GMm}{r_0^2}$ where centripetal motion implies $\frac{GMm}{r_0^2} = \frac{mv^2}{r_0}$, let's rewrite
   $E = -\frac{GMm}{2r_0} = E_n \implies n^2 = r_0\frac{GMm\mu}{\hbar^2} = \frac{r_0}{a_g}$ which with the numbers $n \approx 10^{74}$

(d) Suppose Earth transitioned from n to $n-1$ at the value of n predicted above. What would the wavelength of the emitted excitation be?

We need $E_n - E_{n-1} = -\frac{\mu}{2\hbar^2}(GMm)^2(\frac{1}{n^2} - \frac{1}{(n-1)^2}) \approx \frac{\mu}{2\hbar^2}(GMm)^2\frac{2}{n^3}$

With the above values $\Delta E \approx 10^{-41}[J]$ which implies $\lambda = \frac{hc}{\Delta E} \approx 10^{15}[m]$ which is one lightyear.

2. **Expectations**

Let's compare the Bohr radius to the average groundstate position of the electron.

Recall the groundstate wavefunction for the electron is

$$\Psi_{1,0,0} = R_{10}Y_0^0 = \frac{2}{\sqrt{a}}\frac{1}{a}e^{-\frac{r}{a}}\frac{1}{\sqrt{4\pi}} \tag{1}$$

(a) Compute $\langle \hat{r} \rangle$ in the groundstate and compare to the Bohr radius.

$\langle r \rangle = \int_0^\infty dr \int_0^\pi d\phi \int_0^{2\pi} d\theta \; r^2 \sin\theta \; \Psi^* r \Psi = \frac{4}{a^3}\int_0^\infty dr \; r^3 e^{-\frac{2r}{a}} = \frac{-4}{a^3}\frac{\partial^3}{\partial\alpha^3}[\frac{1}{\alpha}]$ for $\alpha \equiv \frac{2}{a}$
$= \frac{4}{a^3}\frac{6}{\alpha^4} = \frac{3}{2}a$

(b) Compute $\langle \hat{r}^2 \rangle$ and the uncertainty $\sigma^2 \equiv \langle \hat{r}^2 \rangle - \langle \hat{r} \rangle^2$

$\langle r^2 \rangle = \int_0^\infty dr \int_0^\pi d\phi \int_0^{2\pi} d\theta \; r^2 \sin\theta \; \Psi^* r^2 \Psi = 3a^2$ similar to above. $\sigma^2 = \frac{3a^2}{4}$

Hint: Use $\int_0^\infty dr \; r^n \; e^{-\alpha r} = (-1)^n \frac{\partial^n}{\partial\alpha^n}\int_0^\infty dr \; e^{-\alpha r} = (-1)^n \frac{\partial^n}{\partial\alpha^n}[\frac{1}{\alpha}]$