HW 5 Solutions

6.25) $n=2$

$$H_z' = \frac{e}{2m}\left(\vec{L}+2\vec{S}\right)\cdot B_{ext}$$

$$H_{fs}' = -\frac{p^4}{8m^3c^2} + \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}$$

As in the book, I'll use j, m_j, and l as my quantum numbers. J^2, L^2, and J_z commute with each other, and they commute with H_{fs}'.

we have 8 states:

$l=0$

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = |0\ 0\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = |0\ 0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$l=1$

$$\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = |1\,1\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = |1\,{-1}\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{2/3}\,|1\,0\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{1/3}\,|1\,1\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = -\sqrt{1/3}\,|1\,0\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{2/3}\,|1\,1\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = \sqrt{1/3}\,|1\,{-1}\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{2/3}\,|1\,0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = -\sqrt{2/3}\,|1\,{-1}\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{1/3}\,|1\,0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

all H_{fs}' off diagonal elements are zero (pg. 259)

$$-W_{fs} = \begin{pmatrix} 5\gamma & & & & & & & \\ & 5\gamma & & & & & & \\ & & \gamma & & & & & \\ & & & \gamma & & & & \\ & & & & \gamma & & & \\ & & & & & 5\gamma & & \\ & & & & & & \gamma & \\ & & & & & & & 5\gamma \end{pmatrix}$$

for H_z', there are 4 states that are eigenstates of L_z and S_z. Again, only the diagonal elements survive for these states. The other 4 states have diagonal elements

6.25) $\langle H_z' \rangle = \frac{e\hbar}{2m} B_{ext} (m_l + 2m_s)$

$W_{11} = \beta, \; W_{22} = -\beta, \; W_{33} = 2\beta, \; W_{44} = -2\beta$

$W_{55} = \frac{2}{3}\beta, \; W_{66} = \frac{1}{3}\beta, \; W_{77} = -\frac{2}{3}\beta, \; W_{88} = -\frac{1}{3}\beta, \; W_{56} = -\frac{\sqrt{2}}{3}\beta, \; W_{78} = -\frac{\sqrt{2}}{3}\beta$

$$-W_z = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}\beta & -\frac{\sqrt{2}}{3}\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/3\,\beta & \frac{1}{3}\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3}\beta & -\sqrt{2}/3\,\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}/3\,\beta & -\frac{1}{3}\beta \end{pmatrix}$$

6.39) $V = \frac{-e}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i}$ $\qquad r_i$

(a)

just looking at the y direction (since it's all symmetric anyways)

$$V_y = \frac{-eq_2}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(x+d_2)^2+y^2+z^2}} + \frac{1}{\sqrt{(x-d_2)^2+y^2+z^2}} \right]$$

expand $\hookrightarrow (r^2 + d_2^2 + 2xd_2)^{-1/2} = \frac{1}{d_2}\left(\frac{r^2}{d_2^2} + 1 + \frac{2x}{d_2}\right)^{-1/2}$

$\left(1 + \frac{2x}{d_2} + \frac{r^2}{d_2^2}\right)^{-1/2} = (1+\epsilon)^{-1/2} \qquad \epsilon = \frac{2x}{d_2} + \frac{r^2}{d_2^2}$

$\approx 1 - \frac{x}{d_2} - \frac{r^2}{2d_2^2} + \frac{3}{8}\frac{4x^2}{d_2^2}$ $\qquad$ (from the given solution, I know I want to keep terms of order r^2)

6.39) $V_y \approx -\frac{e q_2}{4\pi\epsilon_0 d_2}\left(1 - \cancel{\frac{y}{d_2}} - \frac{r^2}{2d_2^2} + \frac{3}{8}\frac{4x^2}{d_2^2} + 1 + \cancel{\frac{y}{d_2}} - \frac{r^2}{2d_2^2} + \frac{3}{8}\frac{4x^2}{d_2^2}\right)$

$$V_y = -\frac{e q_2}{2\pi \epsilon_0 d_2} - \frac{e q_2}{4\pi\epsilon_0 d_2^3}\left(3x^2 - r^2\right) = \boxed{2\beta_2 d_2^2 + 3\beta_2 x^2 - \beta_2 r^2}$$

it will be the exact same procedure for the other two pairs

(b) $\psi_{gs} = |100\rangle$

$$E' = \langle 100|V_0|100\rangle + 3\langle 100|\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2|100\rangle - (\beta_1 + \beta_2 + \beta_3)\langle 100|r^2|100\rangle$$

ground state is an s-orbital and is therefore spherically symmetric ($\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$)

$$E' = V_0 + \cancel{3(\beta_1 + \beta_2 + \beta_3)\langle 100|x^2|100\rangle - 3(\beta_1 + \beta_2 + \beta_3)\langle 100|x^2|100\rangle}$$

$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle = 3\langle x^2\rangle$

$\boxed{E' = V_0}$ no integrals needed!

if you don't believe me take a look at problem 4.13

9.1) $H' = eEr\cos\theta$

$\psi_1 = |100\rangle$
$\psi_2 = |200\rangle$
$\psi_3 = |210\rangle$
$\psi_4 = |211\rangle$
$\psi_5 = |21\,{-1}\rangle$

$H'_{14} = H'_{15} = 0$ since $\int_0^{2\pi} e^{\pm i\phi}\,d\phi = 0$

$H'_{12} = 0$ since $\int_0^{\pi} \cos\theta\sin\theta\,d\theta = 0$

H'_{13} is the only nonzero transition

$$= \frac{eE}{4\sqrt{2}\,\pi a^4}\int_0^{\infty} r^4 e^{-3r/2a}\,dr \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \boxed{\left(\frac{2^8}{3^5\sqrt{2}}\right) eEa}$$

$H'_{11} = H'_{22} = 0$ $\quad \int_0^{\pi}\cos\theta\sin\theta\,d\theta = 0$

$H'_{33} = H'_{44} = H'_{55}$ $\quad \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta = \int_0^{\pi}\cos\theta\sin^3\theta\,d\theta = 0$

all diagonal elements are zero