Wave Kinetics

$N(\underline{k}, \underline{x}, t)$ → effective distribution function of waves.

Why? →

"A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions"

— Leonardo da Vinci

Codice Atlantico, c 1500.

⇒ Waves come in packets, spectra, etc.!

Wave Adiabatic Theory / Wave Kinetics

– frequently encounter problems with slowly varying parameters $\Rightarrow$ adiabatic theory

$\Rightarrow$

– wave kinetic equation (consequence of Liouville Thm.)

$$\partial_t N + (\underline{V}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k}\cdot\underline{V}) \cdot \partial_{\underline{k}} N$$

$$= C(N)$$; obvious analogy to Boltzmann Eqn.

$N \equiv \mathcal{E}/\omega_{\underline{k}}$ $\equiv$ wave action density / wave quanta density

$\mathcal{E}$ = wave energy density

$$\mathcal{E} = \frac{\partial}{\partial\omega}(\omega\epsilon_k)\Big|_{\omega_{\underline{k}}} \frac{|E_k|^2}{8\pi}, \text{ for e.s.}$$

Characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial\omega}{\partial\underline{k}}\hat{k} + \underline{V}, \qquad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial\underline{x}}(\omega + \underline{k}\cdot\underline{V})$$

refraction by parametric variation (ω); refraction by shear ($\underline{k}\cdot\underline{V}$)

– need:

$$\omega < \frac{1}{\lambda}\frac{d\lambda}{dt} \qquad \lambda \equiv \text{parameter}$$

$$\boxed{\text{space and time scale separation}}$$

$$\frac{1}{N}(\underline{V}_{gr}\cdot\nabla N) \ll \omega \quad \Rightarrow \quad \underline{\nabla}\cdot\underline{V}_{gr} < \omega$$

c(N) → interactions with comparable scale.

Examples:

- linear theory of Langmuir turbulence
  i.e. when will phonon grow?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasmon
  energy → net impact?!
- drift waves and sheared flow.

  ⋮

Fundamentals of wave kinetics

→ where does conservation of action emerge from?

→ answer: phase symmetry (of wave train) underlies wave kinetics

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves" Chapt. 14.

→ Derivation

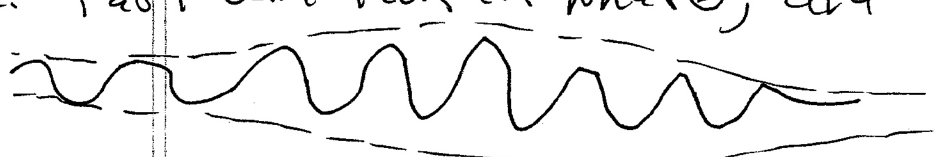
Consider a system, like ideal MHD, which can be described in terms of displacement $\underline{\xi}$:

i.e. $$\underline{\xi} = \mathrm{re}\left\{ A e^{i\phi} + A^{*} e^{-i\phi} \right\}$$

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x} \; \mathcal{L}(\xi)$$

- Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, a la WKB :



$$S = \int dt \int d\underline{x} \; \mathcal{L}(\omega, \underline{k}, a)$$

↑ amplitude

$$\underline{k} = \underline{\nabla}\phi$$
$$\omega = -\partial_t \phi$$

$$= \int dt \int d\underline{x} \; \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

→ neglect all corrections to eikonal theory.

⇒ - here L corresponds to period-averaged Lagrangian
- ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\delta S/\delta a = 0$$

$$\delta S/\delta \phi = 0$$

Now, in linear theory:

$$\left[G(k,\omega) \equiv \frac{\partial G}{\partial \omega} \right]$$

$$\mathcal{L} = G(\omega, k)\, a^2$$

i.e. for MHD, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho \dot{\underline{\xi}}^2 - \frac{1}{2}\rho \left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

↳ eikonal form of stiffness matrix (→ potential energy)

$$\Rightarrow \underline{\xi} \cdot \underline{\underline{M}} \cdot \underline{\xi}$$

$M(k, \omega, \theta)$, as for linear waves

If: $\underline{\xi} = \underline{A} e^{i\phi} + \underline{A}^* e^{-i\phi}$

so $\quad G(\omega, \underline{k}) = \frac{1}{2} \rho \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left[\Omega(\underline{\nabla} \phi, \underline{x}, t) \right]^2 \right]$

Now, 1) $\delta S / \delta a = 0$

$\Rightarrow \quad G(\omega, k) = 0$

but
$$G(\omega, k) = \rho \left(\frac{\partial \phi}{\partial t} \right)^2 - \left[\Omega(\underline{\nabla} \phi, \underline{x}, t) \right]^2$$
$$= \rho \omega^2 - \Omega^2$$

$\Rightarrow$ dispn. relation

2) $\delta S / \delta \phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)} \delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})} \delta(\phi_{\underline{x}}) \right\}$$

end pts fixed, ibp

$$= \int dx \int d^3x \left\{ \partial_t \left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)} \right) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial(\underline{\nabla} \phi)} \right) \right\} \delta \phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)} \right) - \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Now, have: $G(\underline{k}, \omega) = 0$ (dispn. reln.)

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} \cdot d\underline{k} = 0$$

$$\therefore \quad V_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G / \partial \underline{k}}{\partial G / \partial \omega} \qquad \text{(along } \omega \text{)}$$

$\therefore$

$$\partial_t \left((\partial G / \partial \omega) a^2 \right) + \nabla \cdot \left[\frac{-\partial G / \partial \underline{k}}{\partial G / \partial \omega} \frac{\partial G}{\partial \omega} a^2 \right] = 0$$

and so $\quad N \equiv \dfrac{\partial G}{\partial \omega} a^2$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left(\underline{v}_{gr} N \right) = 0$$

(N not $\underline{\underline{\text{yet}}}$ action)

→ Also note energy is conserved ⟺ G invariant to time translations.

so, Noethers thm ⇒ there exists an [redacted] energy conservation equation

have $\mathcal{L} = G(k,\omega)\, a^2$

$$\partial \mathcal{L}/\partial a = 0 \Rightarrow G(\omega, k) = 0$$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

and of course:

$$\underline{\nabla} \times \underline{k} = 0 \quad , \quad \text{as } \underline{k} = \underline{\nabla}\phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad , \text{as} \quad \partial_t \underline{\nabla}\phi = -\underline{\nabla}\left(-\frac{\partial}{\partial t}\phi\right) \checkmark$$

Now, $\mathcal{L} = 0$, as $G(k,\omega) = 0$

as expect $\frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N$, $\quad \omega \frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$

$\Rightarrow 0$, [redacted] creatively

$$\Rightarrow \frac{\partial}{\partial t}\left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L}\right) + \nabla\cdot\left[-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right] = 0$$

$-\frac{\partial G/\partial k}{\partial G/\partial \omega} \frac{\partial G}{\partial \omega} a^2$

[redacted] $\Rightarrow V_{gr}\,\mathcal{E}$

$$\partial_t \left(\omega \mathcal{L}_\omega + \mathcal{L}\right) + \underline{\nabla}\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

check:

$$(\partial_t \omega)\mathcal{L}_\omega + \omega\, \partial_t(\mathcal{L}_\omega) - \partial\mathcal{L}/\partial t$$
$$+ \underline{\nabla}\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$\Rightarrow$ but $\quad \partial_t \mathcal{L}_\omega = \nabla\cdot\left(\mathcal{L}_{\underline{k}}\right)$

$\therefore$

$$(\mathcal{L}_\omega)(\partial_t \omega) + \omega\, \cancel{\nabla\cdot(\mathcal{L}_k)} - \omega\, \cancel{(\nabla\cdot\mathcal{L}_{\underline{k}})}$$
$$- \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right)\cdot\underline{\nabla}\omega - \frac{\partial \mathcal{L}}{\partial t}$$

but $\quad \partial_t \underline{k} = -\underline{\nabla}\omega$

$$(\partial_t \omega)(\mathcal{L}_\omega) + (\partial_t \underline{k})\cdot\frac{\partial \mathcal{L}}{\partial \underline{k}} - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad \checkmark$$

(identity)

$$\Rightarrow \boxed{\partial_t \left\{\omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L}\right\} + \nabla\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0}$$

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\partial \omega} \rightarrow \text{energy density}$$

so

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega \Rightarrow \text{action density } \mathcal{b}$$
$$= N$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \underline{\nabla} N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial\omega/\partial \underline{x} \, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$ (i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{v}_{gr} N \right] = 0$$

Observe:

$\rightarrow$ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

$\rightarrow$ continuity-type equation in $\underline{x}$-space, for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (\underline{v}_{gr} N) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

<u>Now</u> $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, <u>following</u> packet)

(here $\omega = D(\underline{k},\underline{x},t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr}\cdot\nabla\underline{k}$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\underline{k}}{\partial\underline{x}}$$

∴

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial D}{\partial\underline{x}}$$ agree!

→ Now, can convert from N to $\mathcal{E}$

<u>i.e.</u> $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{\text{rays}} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\Big|_{rays} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\Big|_{rays} = 0$$

Now $\displaystyle \frac{d\omega}{dt} = \partial_t \omega + \frac{\partial \omega}{\partial \underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial \omega}{\partial \underline{k}}\cdot\frac{d\underline{k}}{dt}$

From eikonal eqns:

$$= \partial_t \omega + \cancel{\frac{\partial \omega}{\partial \underline{x}}\cdot\frac{\partial \omega}{\partial \underline{k}}} - \cancel{\frac{\partial \omega}{\partial \underline{k}}\cdot\frac{\partial \omega}{\partial \underline{x}}}$$

so if $\partial_t \omega = 0$

$\therefore$ $\displaystyle \frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0.$

$\underline{\underline{so}}$ $\displaystyle \partial_t \mathcal{E} + \underline{V}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial \omega}{\partial \underline{x}}\cdot\underline{\nabla}_{\underline{k}}\mathcal{E} = 0$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t \mathcal{E} + \nabla\cdot\left[\underline{V}_{gr}\,\mathcal{E}\right] = 0}$$

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot \left[\underline{V}_{gr} \mathcal{E} \right] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

$$\Rightarrow \boxed{\underline{\nabla} \cdot \left[\underline{V}_{gr} \mathcal{E} \right] = 0}$$

incompressible wave energy flux!

$\Rightarrow V_{gr}$ drops $\Rightarrow$ $\mathcal{E} \uparrow \Rightarrow$ blocking, breaking

Summary

Recall :

→ Hamiltonian structure of eikonal theory, etc. ⇒

$$\frac{\partial}{\partial t}\rho(\underline{k}, \underline{x}, t) + \underline{v}_{gr}\cdot\underline{\nabla}\,\rho(\underline{k}, \underline{x}, t) - \frac{\partial\omega}{\partial \underline{x}}\cdot\underline{\nabla}_k\,\rho(\underline{k}, \underline{x}, t) = 0$$

→ Physical arguments suggest $\rho = \dfrac{\mathcal{E}}{\omega} = N$ — wave action density

→ Variational Approach

$$S = \int dt \int d^3x\, \mathcal{L}\,, \qquad \mathcal{L} = G(\omega, \underline{k})\, a^2$$

$$\omega = -\partial\phi/\partial t = -\phi_t$$

$$\underline{k} = \underline{\nabla}\phi = \phi_{\underline{x}}$$

$$\delta S = 0$$

but two parameters varied $\left\langle \begin{matrix} a \\ \phi \end{matrix} \right.$

$$\delta S/\delta a = 0 \;\Rightarrow\; G(\omega, \underline{k}) = 0 \;\rightarrow \text{dispersion relation}$$

$$\delta S/\delta\phi = 0 \;\Rightarrow\; \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\omega}\right) - \frac{\partial}{\partial \underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial\omega}a^2\right) - \frac{\partial}{\partial \underline{x}}\cdot\left(\frac{\partial G}{\partial \underline{k}}a^2\right) = 0$$

and time translation symmetry and $G = 0$ ⇒

[redacted]

$$\mathcal{E} = \omega \frac{\partial \epsilon}{\partial \omega} a^2 \quad \Rightarrow \quad N = \frac{\mathcal{E}}{\omega} = \frac{\partial \epsilon}{\partial \omega} a^2$$

and $\quad \frac{\partial \epsilon}{\partial \underline{k}} a^2 = \underline{v}_{gr} \cdot N$

→ Helpful Reminder:

recall, for electrostatic plasma waves

if $\quad \epsilon(\omega, \underline{k}) = 0 \quad \Rightarrow$ dispersion relation

then
$$\mathcal{E}_k = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_{\underline{k}}} \frac{|E_k|^2}{8\pi}$$

$$= \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi} \quad \to \text{wave energy density}$$

$$\therefore \quad N_k = \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

and
$$\underline{\Gamma}_k = -\frac{\partial \epsilon}{\partial \underline{k}}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi} \quad \to \text{wave energy density flux}$$

$$= \underline{v}_{gr} N_{\underline{k}}$$

since $G(\underline{k}, \omega) = 0$, so along rays

$$dG = d\omega \frac{\partial G}{\partial \omega} + d\underline{k} \cdot \frac{\partial G}{\partial \underline{k}} = 0$$

$$d\omega/d\underline{k} = -(\partial G/\partial \underline{k})/(\partial G/\partial \omega)$$

etc.