Physics 4B — Quiz 2 Solutions — Spring 2013

Problem 1

$D(x,t) = 0.1 \sin(6x + 24t) = A\sin(kx + \omega t)$

$A = 0.1, \quad k = 6, \quad \omega = 24.$

$\boxed{\text{Direction: } -x \text{ direction.}}$ $\boxed{\text{Speed: } v = \frac{\omega}{k} = 4\ \text{m/s}}$

(b) Wavelength: $k = \frac{2\pi}{\lambda} \Rightarrow \boxed{\lambda = \frac{2\pi}{k} = 1.05\,\text{m}}$

Frequency: $\omega = 2\pi f \Rightarrow \boxed{f = \frac{\omega}{2\pi} = 3.82\ \text{Hz}}$

Speed: $\frac{\partial D}{\partial t} = \omega A \cos(kx + \omega t)$; $\left.\frac{\partial D}{\partial t}\right)_{max} \equiv v_{max} = \omega A$

$\Rightarrow \boxed{v_{max} = 24 \times 0.1\ \frac{m}{s} = 2.4\ \text{m/s}}$

(c) $v = \sqrt{\frac{F_T}{\mu}} \Rightarrow \mu = \frac{F_T}{v^2}$, $\mu = \text{mass/unit length} = m/l$

$l = 3\,\text{m}, \ v = 4\,\text{m/s}, \ F_T = 3.2\,\text{N} \Rightarrow \boxed{m = \frac{F_T}{v^2} l = \frac{3.2}{4^2} \cdot 3\ \text{kg} = 0.6\ \text{kg}}$

(d) $\boxed{D_{standing}(x,t) = 0.1 \sin(6x)\cos(24t) = A \sin kx \cos \omega t}$

(e) The minimum length is $\boxed{\lambda/2 = \frac{1.05}{2}\,\text{m} = 0.525\,\text{m} = l_{min}}$

$\boxed{\text{Other possible lengths are } n\, l_{min}, \text{ with } n \text{ integer}}$

Problem 2

$$2\frac{dw}{dt} = -w - \frac{d^2w}{dt^2}$$

$w(t) = te^{-\lambda t} \Rightarrow \frac{dw}{dt} = -\lambda te^{-\lambda t} + e^{-\lambda t}$, $\frac{d^2w}{dt^2} = \lambda^2 te^{-\lambda t} - 2\lambda e^{-\lambda t} \Rightarrow$

replacing in diff equation:

$-2\lambda te^{-\lambda t} + 2e^{-\lambda t} = -te^{-\lambda t} - \lambda^2 te^{-\lambda t} + 2\lambda e^{-\lambda t}$; cancel $e^{-\lambda t}$, group $\Rightarrow$

$-2\lambda t + t + \lambda^2 t = -2 + 2\lambda \Rightarrow t(\lambda^2 - 2\lambda + 1) = 2(\lambda - 1) \Rightarrow$

$\Rightarrow t(\lambda-1)^2 = 2(\lambda-1) \Rightarrow \boxed{\lambda = 1}$ so this is valid for all t.

$\Rightarrow \boxed{w(t) = te^{-t}}$ is a solution. No other λ works.

(b) $w(t) = e^{-\alpha t} \Rightarrow \frac{dw}{dt} = -\alpha e^{-\alpha t} \Rightarrow \frac{d^2w}{dt^2} = \alpha^2 e^{-\alpha t} \Rightarrow$ substituting,

$-2\alpha e^{-\alpha t} = -e^{-\alpha t} - \alpha^2 e^{-\alpha t} \Rightarrow \alpha^2 - 2\alpha + 1 = 0 \Rightarrow (\alpha-1)^2 = 0 \Rightarrow \boxed{\alpha = 1}$

$\Rightarrow$ $\boxed{w(t) = e^{-t} \text{ is also a solution}}$

(c) Since $w_1(t) = te^{-t}$ and $w_2(t) = e^{-t}$ are both solutions and since it is a linear differential equation $\Rightarrow w_1(t) + w_2(t)$ is also a solution.

(d) If the factor 2 is replaced by $2a$, the equation in (b) is now

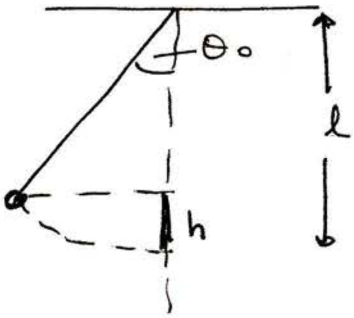
$\alpha^2 - 2a\alpha + 1 = 0$, solving find, $\boxed{\alpha = a \pm \sqrt{a^2 - 1}}$

If $\boxed{a < 1,}$ $\sqrt{a^2-1}$ is imaginary $\Rightarrow \alpha$ is not a real number,

Physically it means the solution will be oscillatory, since

$e^{i\omega t} + e^{-i\omega t} = \cos(\omega t)$. Same situation as in the damped harmonic oscillator. For $a > 1$ the solution is not oscillatory but decays as $e^{-\alpha t}$, with α real (and >0), like overdamped oscillator.

Problem 3



(a) $h = l - l\cos\theta_0 = l(1-\cos\theta_0)$

$U = mgh = mgl(1-\cos\theta_0)$ = initial potential energy.

$m = 100\,kg$, $l = 0.6125\,m$, $g = 9.80\,\frac{m}{s^2}$, $\theta_0 = 0.2\,rads$ $\Rightarrow$ $\boxed{U = 11.97\,J}$

(b) $\theta(t) = \theta_0 \cos\omega t$, $\omega = \sqrt{\frac{g}{l}}$

with $g = 9.80\,m/s^2$, $l = 0.6125\,m$ $\Rightarrow$ $\boxed{\omega = 4\,rad/s}$

in time $t = \frac{\pi}{2\omega}$, $\omega t = \frac{\pi}{2} \Rightarrow \cos\frac{\pi}{2} = 0 \Rightarrow \theta = 0 \Rightarrow$ lowest position

$\Rightarrow$ $\boxed{t = \frac{\pi}{2\omega} = \frac{\pi}{8}\,s = 0.39\,s}$

(c) $\frac{d\theta}{dt} = -\omega\theta_0 \sin(\omega t)$, $v = l\frac{d\theta}{dt}$, at lowest point $\sin(\omega t) = 1$

$\Rightarrow v = \omega\theta_0 l = \frac{4}{s} \times 0.2 \times 0.6125\,m$ $\Rightarrow$ $\boxed{v = 0.49\,\frac{m}{s}}$

Kinetic energy: $K = \frac{1}{2}mv^2 = \frac{1}{2} \times 100 \times 0.49^2\,J$ $\Rightarrow$ $\boxed{K = 12.01\,J}$

(e) We found that K is slightly larger than U, why? Energy is conserved. Reason is, K is not quite right. To derive $\theta(t)$ one approximates $\sin\theta \approx \theta$, which is not exact. $\boxed{\text{U is the better answer}}$

$\boxed{\text{K is slightly incorrect}}$. Note that $\cos\theta_0 \approx 1 - \frac{\theta_0^2}{2} + \frac{\theta_0^4}{4!} \Rightarrow$

$\Rightarrow U = mgl\left(\frac{\theta_0^2}{2} - \frac{\theta_0^4}{4!}\right)$; if we neglect the second term, $(\theta_0^4 \ll \theta_0^2)$

$$\boxed{U = mgl\frac{\theta_0^2}{2} = \frac{m}{2}\omega^2 l^2 \theta_0^2 = \frac{m}{2}v^2 = K}$$