→ Eikonal Theory; Supplement

Recall:

- for Helmholtz Eqn, derived:

$$(\nabla\phi)^2 = \frac{\omega^2}{c(x)^2} \quad \Rightarrow \text{eikonal eqn.}$$

↳ inhomogeneous speed.

$$\psi \sim A e^{i\phi}$$

- as rays $\perp$ phase fronts

$$\underline{\nabla}\phi \equiv \underline{k} = k(\underline{x})$$

old WKB

$$\omega = -\frac{\partial\phi}{\partial t}$$

so

$$\psi = A\exp\left[i\overbrace{\left(\int \underline{k}(\underline{x})\cdot d\underline{x} - \omega t\right)}^{\Phi}\right]$$

eikonal approx. to wave function

- now, for ray trajectories, observe total phase $\underline{\Phi}$

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\, dt$$

$$= \left(\underline{k}\cdot\frac{d\underline{x}}{dt} - \omega\right) dt$$

analogous

$$S = \int L\, dt \Rightarrow dS = L\, dt$$
$$= (\underline{p}\,\dot{\underline{q}} - H)\, dt$$

$\therefore$

– obvious analogy

$\underline{k} \Leftrightarrow \underline{p}$ i.e. QM: $\underline{p} = \hbar \underline{k}$

$\underline{x} \Leftrightarrow \underline{q}$ $E = \hbar\omega$

$\omega \Leftrightarrow H$

so

$$\frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}} \Leftrightarrow \frac{d\underline{p}}{dt} = -\frac{\partial H}{\partial\underline{q}}$$

$$\frac{d\underline{x}}{dt} = \frac{\partial\omega}{\partial\underline{k}} \Leftrightarrow \frac{d\underline{q}}{dt} = \frac{\partial H}{\partial\underline{p}}$$

eg in terms $c(\underline{x})$:

$$\omega^2 = c(x)^2 k^2$$

$$\frac{d\underline{k}}{dt} = -k \frac{\partial c(\underline{x})}{\partial \underline{x}}$$

$$\frac{d\underline{x}}{dt} = c(\underline{x})\,\hat{k}.$$

N.B.

$\rightarrow$ $\partial\omega/\partial\underline{k} \equiv \underline{v}_{gr}$ group velocity

What does $\underline{v}_{gr}$ mean?

Consider wave packet, — 

carrier $\underline{k}_0$

spread $\Delta\underline{k}$

$$\phi\big|_{t=0} \sim \underbrace{e^{i\underline{k}_0\cdot\underline{x}}}_{\text{carrier}}\ \underbrace{F(\underline{x})}_{\text{envelope}}$$

$$F(\underline{x}) \sim \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}}$$

so

$$\phi(\underline{x},t) \sim \sum_{\Delta\underline{k}} e^{i[(\underline{k}_0+\Delta\underline{k})\cdot\underline{x} - \omega(\underline{k}_0+\Delta\underline{k})t]}$$

carrier

$$\sim e^{i(\underline{k}_0\cdot\underline{x} - \omega(\underline{k}_0)t)} \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}}\, e^{-\frac{\partial\omega}{\partial\underline{k}}\cdot\Delta\underline{k}\,t}$$

so $\quad \phi(\underline{x},t) \sim e^{i(\underline{k}_0\cdot\underline{x}-\omega t)} F\Big(\underline{x} - \underbrace{\frac{\partial\omega}{\partial\underline{k}}}_{\underline{v}_{gr}} t\Big)$

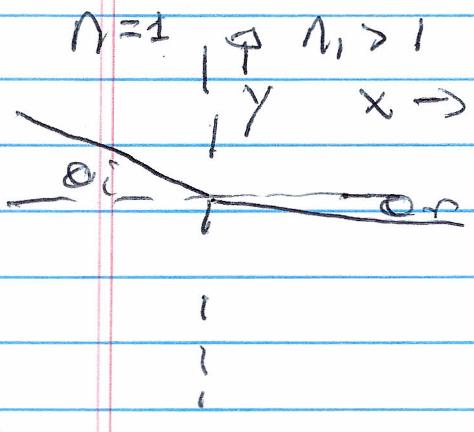
$\Rightarrow$ rate/speed at which energy propagated.

N.B. $\quad E_{pckt} \sim |\phi_k|^2 \sim |F|^2$

$\Rightarrow$ v_{gr} sets speed at which energy propagates.

$\rightarrow \quad \frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}}$ $\qquad \Rightarrow$ Snell's Law.

i.e.



$\frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}} \qquad \Rightarrow \quad \frac{dk_y}{dt} = 0$

$k_{y_-} = k_{y_+} \qquad \Rightarrow \quad k_- \sin\theta_i = k_+ \sin\theta_r$

$k_-^2 = n_0^2 \frac{\omega^2}{c_0^2} \qquad k_+^2 = n_1^2\omega^2/c_0^2$

$$n_0 \sin\theta_i = n_i \sin\theta_r \quad \checkmark.$$

- Now, if seek first principles approach

$\Rightarrow$ extremize Φ (i.e. look for phase stationarity)

$$\delta\Phi = \delta\int\left[\underline{k}\cdot d\underline{x} - \omega dt\right]$$

$$= \delta\int\left[\underline{k}\cdot\dot{\underline{x}} - \omega\right]dt$$

$$= \int\left[\delta\underline{k}\cdot\dot{\underline{x}} + \underline{k}\cdot\delta\dot{\underline{x}} - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]dt$$

but $\delta\dot{\underline{x}} = \frac{d}{dt}\delta\underline{x}$

e.p. fixed.

$$\delta\Phi = \cancel{\underline{k}\cdot\delta\underline{x}\Big|_{t_1}^{t_2}} + \int\left[\delta\underline{k}\cdot\dot{\underline{x}} - \frac{d\underline{k}}{dt}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]$$

$\delta \underline{\Phi} \Rightarrow$

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial \underline{k}} \quad , \quad \frac{dk}{dt} = -\frac{\partial \omega}{\partial \underline{x}} .$$

$\Rightarrow$ Liouville Thm $\rightarrow$ Wave Kinetics.

N.B.: for semi-classical limit

$$\underline{P} = N \hbar \underline{k}$$

$$E = N \hbar \omega$$

$N \Leftrightarrow \rho$.