Solutions (Abbreviated)

1.)

a.) Energy

$$\frac{dL}{dt} = \cancel{\partial_t L} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

LEOM

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \frac{d}{dt}\left[\left(\partial L/\partial \dot{q}\right)\dot{q}\right]$$

$\therefore$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = 0$$

b.) $$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

$$\Psi = \Psi_0 \exp\left[\frac{iS}{\hbar}(x,t)\right]$$

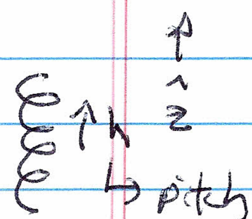
$\hbar \to 0$

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + V$$ $\Rightarrow$ Hamilton–Jacobi Eqn

c.) $$\frac{\partial^2 L}{\partial \dot{\underline{q}}^2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} m$$

$$\det\left[\frac{\partial^2 L}{\partial \dot{\underline{q}}^2}\right] = 0$$ non-invertable for $\dot{q}$ in terms p

$\Rightarrow$ can't construct Hamiltonian

d.) 

$$L_z + \frac{h\,p_z}{2\pi} = \text{const}$$

$$\delta L = \delta z \frac{\partial L}{\partial z} + \delta\phi \frac{\partial L}{\partial \phi} = \delta\phi\left(\frac{h}{2\pi}\frac{\partial L}{\partial z} + \frac{\partial L}{\partial \phi}\right)$$

but $\delta z = \frac{h}{2\pi}\delta\phi$ $$= \delta\phi\left(\frac{h}{2\pi}\dot{p}_z + \dot{L}_z\right)$$

e.) $\underline{\nabla}_\pi \cdot \underline{V}_\pi = 0$

$$\frac{\partial}{\partial \underline{y}} \cdot \frac{dy}{dt} + \frac{\partial}{\partial \underline{x}} \cdot \frac{dx}{dt} = \frac{\partial}{\partial \underline{y}} \cdot \left[\frac{-\partial w}{\partial \underline{x}}\right] + \frac{\partial}{\partial \underline{x}} \cdot \left[\frac{\partial w}{\partial \underline{y}}\right] = 0 \checkmark$$

f.) $$S = \int dt \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - U\right]$$ $U \to \alpha U$

invariance $S \Rightarrow t \sim \left(\sqrt{U}\right)^{-1}$ $\Rightarrow t \to \alpha^{-1/2} t$

g.) Can't rescale U, which is essential to Virial Theorem.

h.) $H = E$

$$\frac{1}{2m}\left[(\partial_r S)^2 + \frac{1}{r^2}(\partial_\phi S)^2 + (\partial_z S)^2\right] + V = E$$

need $V = a(r) + \frac{b(\phi)}{r^2} + c(z)$

i.) <u>No</u>.

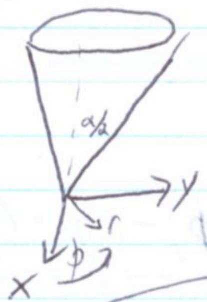
Hamiltonian systems $\Rightarrow \underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = 0$

Attractor in phase space $\Rightarrow \underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma < 0$ in neighborhood of attractor.

j.) α_1/α_2 irrational
$t \to \infty$, trajectory fills toroidal surface. Reason is ergodic thm, related to Poincare recurrence $\Rightarrow$ eventually, trajectory will come arbitrarily close to itself, etc. For α_1/α_2 rational, trajectory closes on self.

Problem 2:

b) Coordinate system 1:



$$\begin{cases} z = \dfrac{r}{\tan(\alpha/2)} \\ x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\Longrightarrow \dot{z} = \frac{\dot{r}}{\tan(\alpha/2)}, \quad \dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$
$$\dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

$$\Longrightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$
$$= \tfrac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{r}^2 + \frac{\dot{r}^2}{\tan^2(\alpha/2)}\right) - \frac{mgr}{\tan(\alpha/2)}$$
$$= \tfrac{1}{2}m\left(\dot{r}^2\left(1 + \frac{1}{\tan^2(\alpha/2)}\right) + r^2\dot{\phi}^2\right) - mgr\cot(\alpha/2)$$

$$\boxed{\Longrightarrow \mathcal{L} = \tfrac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{r}^2\csc^2(\alpha/2)\right) - mgr\cot(\alpha/2)}$$

EoM: $\dfrac{\partial \mathcal{L}}{\partial q^i} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)$

r: $\Longrightarrow \boxed{m\ddot{r} = -mg\sin(\alpha/2)\cos(\alpha/2) + mr\sin^2(\alpha/2)\dot{\phi}^2}$

ϕ: $\Longrightarrow \boxed{mr^2\dot{\phi} = \ell = \text{const}}$

Coordinate system 2:

$$\begin{cases} z = \cos(\alpha/2)\, r \\ x = \sin(\alpha/2)\, r \cos\phi \\ y = \sin(\alpha/2)\, r \sin\phi \end{cases}$$

$$\Rightarrow \underbrace{\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2(\alpha/2)\, \dot{\phi} \right) - mg\cos(\alpha/2)\, r}$$

Coordinate system 3:

$$\begin{cases} z = z \\ x = z\tan(\alpha/2) \cos\phi \\ y = z\tan(\alpha/2)\sin\phi \end{cases}$$

$$\Rightarrow \underbrace{\mathcal{L} = \frac{1}{2} m \left(\dot{z}^2 \sec^2(\alpha/2) + z^2 \tan^2(\alpha/2)\, \dot{\phi}^2 \right) - mgz}$$

c) Using generalized coordinates (1):

Eliminate $\dot{\phi} \longrightarrow \dot{\phi} = \frac{\ell}{mr^2}$

$$\Rightarrow (r \text{ EoM}) \quad \underbrace{m\ddot{r} = -mg\sin(\alpha/2)\cos(\alpha/2) + \frac{\ell^2 \sin^2(\alpha/2)}{mr^3}}$$

Note: This can be used to definitively answer part a)

equilibrium circular orbits, $\ddot{r} = 0$, $r = r_0$

$$\Rightarrow mg\sin(\alpha/2)\cos(\alpha/2) = \frac{\ell^2 \sin^2(\alpha/2)}{r_0^3}$$

$$\cancel{\Rightarrow r_0^3 = \frac{\ell^2 \tan(\alpha/2)}{m^2 g}}$$

$$\Rightarrow r_0^3 = \frac{\ell^2 \tan(\alpha/2)}{m^2 g}$$

(see w at end)

[Alternatively, the existence of equilibrium pts and their stability/distinguishing characteristics could have been surmised apriori based on the system...]

Continuing c), we have:

let $r = r_0 + \delta r$

$$\Rightarrow \delta\ddot{r} = -g\sin(\alpha/2)\cos(\alpha/2) + \frac{\ell^2 \sin^2(\alpha/2)}{m(r_0+\delta r)^3}$$

$$= -g\sin(\alpha/2)\cos(\alpha/2) + \frac{\ell^2 \sin^2(\alpha/2)}{m^2 r_0^3\left(1+\frac{\delta r}{r_0}\right)^3}$$

(Taylor Expand) $\Rightarrow$ $$\delta\ddot{r} \cong -g\sin(\alpha/2)\cos(\alpha/2) + \frac{\ell^2 \sin^2(\alpha/2)}{m^2 r_0^3}\left(1 - 3\frac{\delta r}{r_0}\right)$$

$$\Rightarrow \delta\ddot{r} = -\frac{3\ell^2 \sin^2(\alpha/2)}{m^2 r_0^3 (r_0)}\delta r$$

$$\Rightarrow \delta\ddot{r} = -3\frac{g}{r_0}\sin(\alpha/2)\cos(\alpha/2)\,\delta r$$

$$\boxed{\omega^2 = \frac{3g}{r_0}\sin(\alpha/2)\cos(\alpha/2)}$$

Problem 3

A physical system has kinetic energy

$$T = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right),$$

and potential energy

$$U = \frac{\alpha}{\left(q_1^2 + q_2^2\right)}.$$

(a) Derive the Hamiltonian and the Hamiltonian equations of motion for this system. $H(q,p) = \sum p\dot{q} - L(q,\dot{q})$ where $L = T - U$.

$$p_i = m\dot{q}_i \left(q_1^2 + q_2^2\right), \quad \text{for } i = 1, 2$$

So the Hamiltonian is

$$H = \frac{1}{\left(q_1^2 + q_2^2\right)} \left[\frac{p_1^2 + p_2^2}{2m} + \alpha\right].$$

Hamilton's equations of motion are

$$\dot{q}_i = \frac{1}{\left(q_1^2 + q_2^2\right)} \frac{p_i}{m},$$

$$\dot{p}_i = \frac{2q_i}{\left(q_1^2 + q_2^2\right)} \left[\frac{p_1^2 + p_2^2}{2m} + \alpha\right].$$

(b) Give an explicit expression for the phase space flow and Liouville equation for this system.

$$\vec{v} = (\dot{q}_i, \dot{p}_i), \quad \text{phase space flow}$$

$$0 = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho, \quad \text{Liouville equation}$$

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{\left(q_1^2 + q_2^2\right)} \left[\frac{p_i}{m}\frac{\partial}{\partial q_i} + \left(\frac{p_1^2 + p_2^2}{m} + 2\alpha\right) q_i \frac{\partial}{\partial p_i}\right] \rho$$

(c) Derive the Hamilton-Jacobi equation for this system. The Hamilton-Jacobi equation is

$$H\left(q_i; \frac{\partial S}{\partial q_i}\right) + \frac{\partial S}{\partial t} = 0,$$

and since H does not depend explicitly on time we may take $S = W - Et$. Subtituting H into the above equation yields

$$E = \frac{1}{\left(q_1^2 + q_2^2\right)} \left[\frac{1}{2m}\left(\left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2\right) + \alpha\right].$$

Alternative Solution The $q_1^2 + q_2^2$ terms in the Hamiltonian are suggestive of a rotational symmetry. Make the following coordinate transformation:

$$q_1 = 2\sqrt{r}\cos\left(\frac{\theta}{2}\right),$$
$$q_2 = 2\sqrt{r}\sin\left(\frac{\theta}{2}\right).$$

With these new coordinates r, θ, the Hamiltonian becomes

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\beta}{r},$$

where $\beta = \alpha/4$. Thus, this problem is equivalent to the two-dimensional Kepler problem.

4.) a.) For path, see pgs 9-13 on
eqn.
notes "Hamilton-Jacobi I".
Derivation follows from Abbreviated Action.

b.) See Problem and Solution on Pg. 16
of Landau & Lifshitz text.