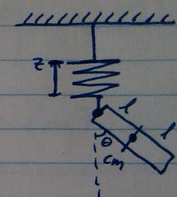
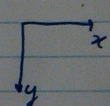
Problem 1



$I = \frac{2^2}{12} m l^2 = \frac{1}{3} m l^2$



$x = l\sin\theta \qquad y = l\cos\theta + z$

$\dot{x} = l\dot{\theta}\cos\theta \qquad \dot{y} = -l\sin\theta\dot{\theta} + \dot{z}$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$$

$$T = \frac{1}{2} m (\dot{z}^2 + l^2\dot{\theta}^2 - 2l\dot{z}\sin\theta\dot{\theta}) + \frac{1}{2} I \dot{\theta}^2$$

$$V = -mg(z + l\cos\theta) + \frac{1}{2} k z^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = \underline{m\dot{z} - \frac{2l\sin\theta\dot{\theta}m}{2}}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\dot{z}\sin\theta + \frac{1}{3} m l^2 \dot{\theta}$$

$$= \underline{\frac{4}{3} m l^2 \dot{\theta} - ml\dot{z}\sin\theta}$$

$$H = p_z\dot{z} + p_\theta\dot{\theta} - \frac{1}{2} m (\dot{z}^2 + l^2\dot{\theta}^2 - 2l\dot{z}\sin\theta\dot{\theta}) - \overbrace{\frac{1}{2} I \dot{\theta}^2}^{\frac{1}{2}m\cdot\frac{1}{3}l^2\dot{\theta}^2} + mg(z + l\cos\theta) + \frac{1}{2} k z^2$$

$$= \underline{\frac{4}{3} m l^2 \dot{\theta} - ml\dot{z}\sin\theta}$$

$$H = p_z\dot{z} + p_\theta\dot{\theta} - \frac{1}{2} m (\dot{z}^2 + l^2\dot{\theta}^2 - 2l\dot{z}\sin\theta\dot{\theta}) - \overbrace{\frac{1}{2} I \dot{\theta}^2}^{\frac{1}{2}m\cdot\frac{1}{3}l^2\dot{\theta}^2} + mg(z + l\cos\theta) + \frac{1}{2} k z^2$$

$$= p_z\dot{z} + p_\theta\dot{\theta} - \frac{1}{2} m \left(\dot{z}(\underbrace{\dot{z} - l\sin\theta\dot{\theta}}_{p_z/m}) + \dot{\theta}(\underbrace{\tfrac{4}{3} l^2\dot{\theta} - l\dot{z}\sin\theta}_{p_\theta/m}) \right) + mg(z + l\cos\theta) + \frac{1}{2} k z^2$$

$$= \boxed{\frac{1}{2} p_z\dot{z} + \frac{1}{2} p_\theta\dot{\theta} - mg(z + l\cos\theta)} + \frac{1}{2} k z^2$$

Find $\dot{z}, \dot{\theta}$

$$\dot{z} = \frac{1}{m}(p_z + ml\sin\theta\dot{\theta}), \quad \dot{\theta} = \frac{1}{4I} * (p_\theta + ml\dot{z}\sin\theta)$$

$$\dot{z} = \frac{1}{m}\left(p_z + \frac{ml\sin\theta}{\frac{4}{3}ml^2}(p_\theta + ml\dot{z}\sin\theta)\right) = \frac{p_z}{m} + \frac{3}{4}\frac{\sin\theta}{ml}p_\theta + \frac{3}{4}\dot{z}\sin^2\theta$$

$$\Rightarrow \dot{z}\left(1 - \frac{3}{4}\sin^2\theta\right) = \underline{\frac{1}{m}p_z + \frac{3}{4}p_\theta \cdot \frac{\sin\theta}{ml}}$$

$$\dot{z} = \frac{1}{m} * \frac{4p_z + 3p_\theta\sin\theta/l}{4 - 3\sin^2\theta}$$

$$\dot{\theta} = \frac{3}{4ml^2}\left[p_\theta + \frac{ml\sin\theta}{m}\left(\frac{4p_z + 3p_\theta\sin\theta/l}{4 - 3\sin^2\theta}\right)\right] = \frac{3p_\theta}{4ml^2} + \frac{3\sin\theta}{4lm}\left(\frac{4p_z + 3p_\theta\sin\theta/l}{4 - 3\sin^2\theta}\right)$$

$$= \underline{\frac{3}{l^2 m}\,\frac{p_\theta + p_z l\sin\theta}{4 - 3\sin^2\theta}}$$

$$H = \frac{1}{2}\frac{p_z}{ml}\left(\frac{4p_z l^2 + 3p_\theta \sin\theta\, l}{4-3\sin^2\theta}\right) + \frac{p_\theta}{2ml^2}\left(\frac{3p_\theta + 3p_z l\sin\theta}{4-3\sin^2\theta}\right) - mg(z + l\cos\theta) + \frac{1}{2}kz^2$$

$$\boxed{H = \frac{1}{2}\cdot\frac{1}{ml^2}\cdot\left(\frac{4p_z^2 l^2 + 3p_\theta^2 + 6p_\theta p_z l\sin\theta}{4-3\sin^2\theta}\right) - mg(z + l\cos\theta) + \frac{1}{2}kz^2}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = \underline{mg - kz}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{3}{ml^2} * \frac{4p_z^2 l^2\cos^2\overbrace{\theta}^{\sin\theta} + 3p_\theta^2\sin\theta\cos\theta + 3p_\theta p_z l\cos\theta\sin^2\theta + 4p_\theta p_z l\cos\theta}{(4-3\sin^2\theta)^2}$$

$$= \underline{-\frac{3}{ml^2}\,\frac{(4p_z^2 l^2 + 3p_\theta^2)\sin\theta\cos\theta + p_\theta p_z l\cos\theta(3\sin^2\theta + 4)}{(4-3\sin^2\theta)^2}}$$

Alex Meill

2)

$$\mathcal{L}(r,\theta,\phi,\dot r,\dot\theta,\dot\phi) = \frac{m}{2}\left[\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right] - V(r)$$

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot r} = m\dot r \rightarrow \qquad \dot r = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot\theta} = mr^2\dot\theta \longrightarrow \qquad \dot\theta = \frac{p_\theta}{mr^2}$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = mr^2\sin^2\theta\,\dot\phi \rightarrow \qquad \dot\phi = \frac{p_\phi}{mr^2\sin^2\theta}$$

$$\mathcal{H} = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} - \frac{m}{2}\left[\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2r^2} + \frac{p_\phi^2}{m^2r^2\sin^2\theta}\right] + V(r)$$

$$\boxed{\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r)}$$

$$\frac{\partial\mathcal{H}}{\partial p_r} = \boxed{\dot r = \frac{p_r}{m}} \qquad -\frac{\partial\mathcal{H}}{\partial r} = \boxed{\dot p_r = -\frac{\partial V}{\partial r} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta}}$$

$$\frac{\partial\mathcal{H}}{\partial p_\theta} = \boxed{\dot\theta = \frac{p_\theta}{mr^2}} \qquad -\frac{\partial\mathcal{H}}{\partial\theta} = \boxed{\dot p_\theta = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta}}$$

$$\frac{\partial\mathcal{H}}{\partial p_\phi} = \boxed{\dot\phi = \frac{p_\phi}{mr^2\sin^2\theta}} \qquad -\frac{\partial\mathcal{H}}{\partial\phi} = \boxed{\dot p_\phi = 0}$$

Job Ir

3. $dl^2 = v^2 = r^2\dot\theta^2 + r^2\sin^2\theta\dot\phi^2$

$T = \frac{1}{2}m r^2(\dot\theta^2 + \sin^2\theta\dot\phi^2)$

$U = -mgl\cos\theta$

$L = \frac{1}{2}m r^2(\dot\theta^2 + \sin^2\theta\dot\phi^2) + mgl\cos\theta \;\Big]\; r = l$

$p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta \qquad p_\phi = \frac{\partial L}{\partial\dot\phi} = ml^2\sin^2\theta\dot\phi$

$$\Rightarrow H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

$\frac{\partial H}{\partial p_i} = \dot q_i$ and $\dot p_i = -\frac{\partial H}{\partial q_i}$

$$\Rightarrow \dot p_\theta = \frac{p_\phi^2\cos\theta}{\sin^3\theta\, ml^2} + mgl\sin\theta$$

$\dot p_\phi = 0 \Rightarrow p_\phi = \text{const} = L$

$\Big[\; \dot\theta = \quad \dot\phi =$

solve for L, for $\theta = \theta_0$

$$\Rightarrow \dot p_{\theta_0} = 0 = \frac{L^2\cos\theta_0}{\sin^3\theta_0\, ml^2} + mgl\sin\theta_0$$

$$L^2 = \frac{m^2 l^3\sin^4\theta_0\, g}{\cos\theta_0}$$

expand Hamiltonian, $\theta \to \theta_0 + \eta$

(sm. angle approx)

$$\cos(\theta_0+\eta) = \cos\theta_0\cos\eta - \sin\theta_0\sin\eta \approx \cos\theta_0 - \frac{\eta^2}{2}\cos\theta_0 - \eta\sin\theta_0$$

$$\sin^2(\theta_0+\eta) = (\sin\theta_0\cos\eta + \sin\eta\cos\theta_0)^2$$

$$= \sin^2\theta_0 - \eta^2\sin^2\theta_0 + \eta\cos\theta_0\sin\theta_0 + \frac{\eta^4}{4}\sin^2\theta_0 - \frac{\eta^3}{2}\sin\theta_0\cos\theta_0$$

$$+ \sin\theta_0\,\eta\cos\theta_0 - \frac{\eta^3}{2}\cos\theta_0\sin\theta_0 + \eta^2\cos\theta_0$$

$$= \sin^2\theta_0 - \eta^2 \sin^2\theta_0 + 2\eta \sin\theta_0 \cos\theta_0 + \eta^2 \cos\theta_0$$

Simplify $\Rightarrow$

$$\frac{1}{\sin^2(\theta_0+\eta)} = \frac{1}{\sin^2\theta_0}\left[1 - 2\eta \cot(\theta_0) + \eta^2\left(1 + 3\cot^3\theta_0\right)\right]$$

ℓ^2

$$H = \frac{P_\theta^2}{2m\ell^2} + \left[\frac{m^2\ell^3\sin^4\theta_0 g}{\cos\theta_0}\right]\left[\frac{1}{2m\ell^2\sin^2\theta_0}\right]\left[1 - 2\eta\cot\theta_0 + \eta^2 + 3\eta^2\cot^3\theta_0\right]$$

$$- m\ell g\left[\cos\theta_0 - \frac{\eta^2}{2}\cos\theta_0 - \eta\sin\theta_0\right]$$

$$\dot{P}_\theta = m\ell^2\ddot{\theta} = m\ell^2\ddot{\eta} = \frac{\partial H}{\partial \eta}$$

$$\frac{\partial H}{\partial \eta} = m\ell^2\ddot{\eta} = \left[\frac{m^2\ell^3\sin^4\theta_0 g}{\cos\theta_0}\right]\left[\frac{1}{2m\ell^2\sin^2\theta_0}\right]\left[-2\cot\theta_0 + 2\eta + 6\eta\cot^3\theta_0\right]$$

$$- m\ell g\left[-\eta\cos\theta_0 - \sin\theta_0\right]$$

terms cancel $\Rightarrow$

$$\ddot{\eta} = -\eta\frac{g}{\ell\cos\theta_0}\left(1 + 3\cos^2\theta_0\right)$$

$$\Rightarrow \omega = \sqrt{\frac{g(1+3\cos^2\theta_0)}{\ell\cos\theta_0}}$$

Mechanics Pset 3, #4 : FW 6.4

a) Lagrangian for relativistic particle in static potential $V(\vec{r})$?

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} - V(\vec{r})$$

Lagrange's equations: $\frac{\partial \mathcal{L}}{\partial \vec{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \rightarrow \frac{d}{dt}\left(\frac{-mc^2\frac{1}{2}\left(-2\frac{\dot{\vec{r}}}{c^2}\right)}{\sqrt{1-v^2/c^2}}\right) = \underbrace{\frac{d}{dt}\left(\frac{m\dot{\vec{r}}}{\sqrt{1-v^2/c^2}}\right) = -\frac{\partial V}{\partial \vec{r}}}_{\star}$

$(\vec{v} = \dot{\vec{r}})$

Solving: $\frac{m\ddot{r}}{\sqrt{1-\dot{r}^2/c^2}}\left(1 + \frac{\dot{r}^2}{c^2}\frac{1}{(1-\dot{r}^2/c^2)}\right) = -\frac{\partial V}{\partial r}$

$$\boxed{\frac{m\ddot{r}}{\left(1-\frac{\dot{r}^2}{c^2}\right)^{3/2}} = -\frac{\partial V}{\partial \vec{r}}}$$

really means $\underbrace{-\vec{\nabla}\cdot V}$

recall that $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \rightarrow \star$ is $\frac{d}{dt}(m\vec{v}\gamma) = -\frac{\partial V}{\partial \vec{r}}$

is $\underbrace{\frac{d}{dt}P_{rel} = F_{rel}}$

$\Rightarrow$ this Lagrangian is indeed that of a relativistic particle.

b) $\underbrace{p = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\gamma\dot{\vec{r}}}$, as shown above.

$$H = p\cdot\dot{q} - \mathcal{L} = m\gamma\dot{r}^2 + \frac{mc^2}{\gamma} + V(\vec{r}) = m\left(\frac{\dot{r}^2}{\sqrt{1-v^2/c^2}} + c^2\sqrt{1-v^2/c^2}\right) + V(\vec{r})$$

$$= \frac{m}{\sqrt{1-v^2/c^2}}\left(\cancel{\dot{r}^2} + c^2 - \cancel{\dot{r}^2}\right) = \underbrace{\frac{mc^2}{\sqrt{1-v^2/c^2}} + V(\vec{r}) = H}$$

We can rewrite this as:

$$H = \frac{mc^2}{\sqrt{1-v^2/c^2}} + V(r) = \frac{mc^2}{\sqrt{1-v^2/c^2}}\sqrt{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}} + V(\vec{r}) = mc^2\sqrt{\frac{c^2(1-v^2/c^2)+v^2}{c^2(1-v^2/c^2)}} + V(\vec{r})$$

$$= mc^2\sqrt{1+\gamma^2v^2/c^2} + V(r) = mc^2\sqrt{1+p^2/m^2c^2} + V(\vec{r}) = \sqrt{m^2c^4+p^2c^2} + V(\vec{r})$$

$$\rightarrow \boxed{H = \sqrt{m^2c^4+p^2c^2} + V(\vec{r})}$$

Note we can be more explicit about finding the momenta :

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \gamma m \dot{r} = p_r$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \gamma m r^2 \dot{\theta} = p_\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \gamma m r^2 \sin^2\theta\,\dot{\phi} = p_\phi$$

$\Rightarrow p_{rel} = m\gamma\dot{\vec{r}}$ where $\dot{\vec{r}}$ has components $(\dot{r},\ r^2\dot{\theta},\ r^2\sin^2\theta\,\dot{\phi})$

$$p_{rel}^2 = m^2\gamma^2(\dot{\vec{r}}\cdot\dot{\vec{r}}) = m^2\gamma^2\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

H is a constant of motion if $\frac{dH}{dt} = 0$:

$$\frac{dH}{dt} = mc^2\frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} + \frac{dV(\vec{r})}{dt} = (\text{componentwise})\ \frac{mv\dot{v}}{(1-v^2/c^2)^{3/2}} + v\frac{\partial V}{\partial r}$$

$$= v\left[\frac{\partial V}{\partial r} + \frac{m\dot{v}}{(1-v^2/c^2)^{3/2}}\right]$$

we already showed e.o.m is $-\nabla\cdot V = \frac{m\dot{v}}{(1-v^2/c^2)^{3/2}}$

$$\Rightarrow \boxed{\frac{dH}{dt} = 0}$$

c) Let V be spherically symmetric.

Then $\frac{d}{dt}(\vec{r}\times\vec{p}) = \left(\frac{d\vec{r}}{dt}\times\vec{p}\right) + \left(\vec{r}\times\frac{d\vec{p}}{dt}\right)$

$$= \underbrace{\left(\vec{v}\times(m\gamma\vec{v})\right)}_{=0} + \underbrace{(\vec{r}\times\vec{F})}$$

$\vec{F}$ is radial for V spherically symmetric, $\to \vec{r}\times\vec{F} = 0$.

$\to \frac{d}{dt}(\vec{r}\times\vec{p}) = 0 \to \underbrace{\vec{r}\times\vec{p} \text{ is a constant of motion.}}$

Of course this makes sense, since angular momentum is conserved for spherically symmetric systems.

Note also that we have $\frac{\partial \mathcal{L}}{\partial \varphi} = 0 = \dot{p}_\varphi \to \mathcal{L}$ is cyclic in φ

$$\to \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = p_\varphi = \gamma m r^2 \sin^2\theta\, \dot{\varphi} = \text{const}$$

p_φ is actually the z-comp of angular momentum!

$$p_\varphi = (\vec{r}\times\vec{p})_z = \vec{r}\times(\gamma m r^2 \sin^2\theta\, \dot{\varphi})$$
$$= \gamma m r^2 \sin^2\theta\, \dot{\varphi}$$

Note that $V_{eff} = \frac{p_\varphi^2}{r^2} = \frac{p_\varphi \cdot p_\varphi}{r^2} = \frac{1}{r^2\sin^2\theta}\gamma^2 m^2 r^4 \sin^4\theta\, \dot{\varphi}^2$

will cancel with this same term in p^2, ~~get~~

reducing the problem to 2 variables

(which should be the case, since for V spherically symmetric we can eliminate a degree of freedom)

$\to$ We can write H as

$$\underbrace{H = c\left(m^2c^2 + p^2 + r^{-2}p_\varphi^2\right)^{1/2} + V(r)}$$

Derive Hamiltonian EOM from modified principle of least Action

Daniel Ben-Zion

$$\mathcal{H} \equiv \sum_i p_i \dot{q}_i - \mathcal{L} \Rightarrow \mathcal{L} = \sum_i p_i \dot{q}_i - \mathcal{H}$$

$$S = \int_1^2 \mathcal{L}\, dt \Rightarrow \int_1^2 \sum_i p_i \dot{q}_i - \mathcal{H}$$ & require $\delta S = 0$

$$\delta S = \delta \int_1^2 \sum p_i \dot{q}_i - \mathcal{H} = \int_1^2 \delta\left[\sum p_i \dot{q}_i - \mathcal{H}\right]$$

$$= \int_1^2 \sum \dot{q}_i \delta p_i + \underbrace{p_i \delta \dot{q}_i}_{p_i \frac{d}{dt}\delta q_i} - \frac{\partial \mathcal{H}}{\partial p_i}\delta p_i - \frac{\partial \mathcal{H}}{\partial q_i}\delta q_i$$

integrate this term by parts to get $\underbrace{\cancel{p_i \delta q_i \Big|_1^2}}_{0 \text{ by construction}} - \int_1^2 \dot{p}_i \delta q_i$

$$\Rightarrow \int_1^2 \sum \underbrace{\left(\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i}\right)}_{\text{must } = 0}\delta p_i - \underbrace{\left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i}\right)}_{\text{must } = 0}\delta q_i = 0$$

p_i & q_i are indep so

$$\Rightarrow \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

6.) $\nabla^2\psi + \frac{\omega^2}{c_0^2} n^2 \psi = 0$

plane wave

a.) let $n^2 = 1 + \delta(\vec{r})$ and $\psi = A(\vec{r}) e^{ikz}$

$\nabla^2 = \left(\frac{\partial^2}{\partial z^2} + \nabla_\perp^2\right)$

$\nabla^2\psi = \nabla_\perp^2\psi + \frac{\partial^2}{\partial z^2}\left[A(r) e^{ikz}\right]$

$= \partial/\partial z \left[\frac{\partial A}{\partial z} e^{-ikz} + ik A e^{ikz}\right]$

$= \frac{\partial^2 A}{\partial z^2} e^{-ikz} + \frac{ik\,\partial A}{\partial z} e^{-ikz} + \left[\frac{ik\,\partial A}{\partial z} e^{ikz} + k^2 A e^{ikz}\right.$

helmholtz

$\nabla_\perp^2\psi + \left[\frac{\partial^2 A}{\partial z^2} + 2ik\left(\frac{\partial A}{\partial z}\right) - k^2 A\right] e^{ikz} + \frac{\omega^2}{c_0^2}\psi + \frac{\omega^2}{c_0^2}\delta(\vec{r})\psi = 0$

if $k = \omega/c$ and $\partial^2 A/\partial z^2 \to 0$ for time independent

just small compared to other terms

$e^{ikz} \boxed{\left[2ik\frac{\partial A}{\partial z} + \nabla_\perp^2 A + \delta(x)\,\omega^2/c_0^2 A\right] = 0}$ $A = |\psi|$

b.) $k_z = \omega/c_0$ since $\delta \ll 1$ A varies slowly compared to other terms $\therefore \frac{\partial^2 A}{\partial z^2} = 0$

$\underbrace{2ik\frac{\partial A}{\partial z}}_{\text{source}} + \underbrace{\nabla_\perp^2 A}_{\text{diffraction}} + \underbrace{\delta(x)\,\omega^2/c_0^2 A}_{\text{scattering}}$ iii)?

c.) $\psi = A(\vec{x}) e^{i\phi(x)}$

$\nabla_\perp^2\psi = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\psi\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\psi\right) = \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + 2i\left[\frac{\partial A}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial \phi}{\partial y}\right] - A\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] + iA\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\right) e^{i\phi}$

$$\frac{\partial \psi}{\partial z} = \left[\frac{\partial A}{\partial z} + i\frac{\partial \phi}{\partial z} A\right] e^{i\phi}$$

$\downarrow$ plug into helmoltz divide by $e^{i\phi}$

$$0 = 2ik\left[\frac{\partial A}{\partial z} + i\frac{\partial \phi}{\partial z} A\right] + \nabla_\perp^2 A + 2i(\nabla A)_\perp \cdot (\nabla\phi)_\perp$$
$$- A\nabla_\perp^2 \phi + iA(\nabla\phi)_\perp^2 + \omega^2/c_0^2\, \delta(x)\, A$$

imag $\left\{ 2k_z \frac{\partial A}{\partial z} + 2(\vec{\nabla} A)_\perp \cdot (\vec{\nabla}\phi)_\perp + A(\vec{\nabla}\phi)_\perp^2 = 0 \right.$

real $\left. -2k_z A \frac{\partial \phi}{\partial z} + \nabla_\perp^2 A - A(\nabla_\perp^2 \phi) + \omega^2/c_0^2\, \delta(x)\, A = 0 \right.$

<u>Eikonal eq.</u>

$$[2i\,\vec{\nabla}A \cdot \vec{\nabla}\phi] - [A(\vec{\nabla}\phi)^2] + \nabla^2 A - [iA\nabla^2\phi] = -\frac{\omega^2}{c_0^2}(1+\delta)A$$

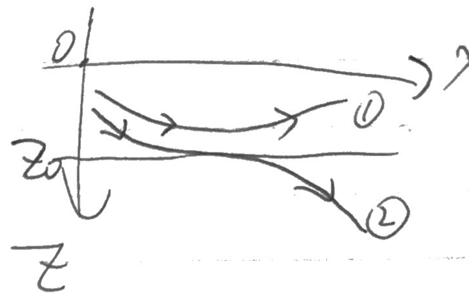
elaborate on this...

involves keeping or throwing out ω^2/c_0^2, no assumption of geometry

Helmholtz - geometry assumption

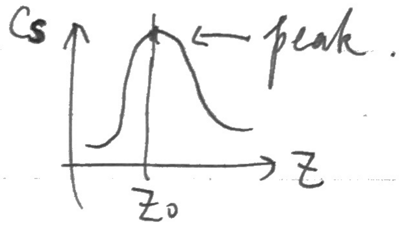
Xiang Fan.

The ocean problem:



$$\begin{cases} \ddot{x} = \dfrac{\partial \ln C_s(z)}{\partial z}\, \dot{z}\dot{x}. \\ \ddot{z} = \dfrac{\partial \ln C_s(z)}{\partial z} \cdot (\dot{z}^2 - 1). \end{cases}$$

if:



Note that $\dot{z} \equiv \frac{dz}{ds} < 1$ always.

$\therefore$ $(\dot{z}^2 - 1)$ always negative.

$\therefore$ when $z < z_0$, $\frac{\partial \ln C_s(z)}{\partial z}$ positive.

near z_0, very large positive.

$\therefore$ $\ddot{z}$ will be very negative.

which means $\dot{z}$ drops very fast.

It might/might not goes negative.

if negative, ①. since $\ddot{z}$ always negative, $|\dot{z}|$ will get larger & larger

if positive until $z = z_0$, $\frac{\partial \ln C_s(z)}{\partial z}$ changes sign,

$\ddot{z}$ will be positive,

$\dot{z}$ will goes larger and larger

which is ②.