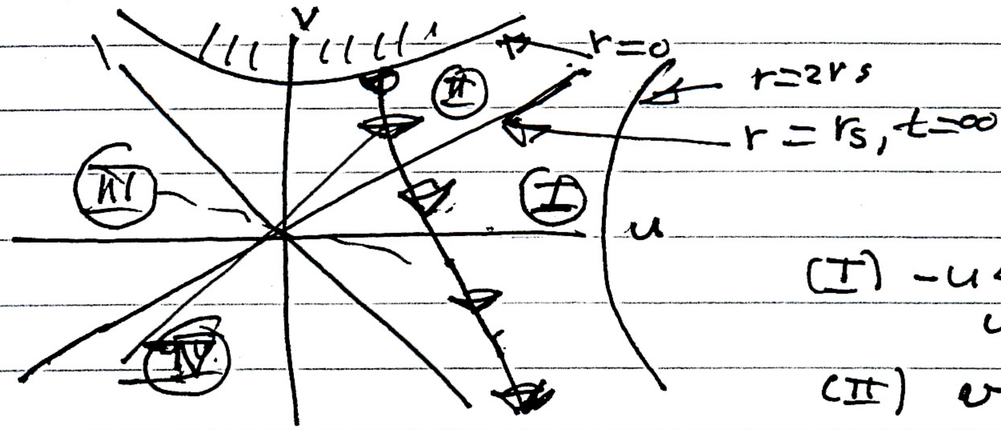
Kruskal/Szekeres extension to Schwarzschild spacetime

Recall: traded $r, t \to u, v$

$$\boxed{ds^2 = -\frac{32\,(GM/c^2)^3}{r}\, e^{-\frac{r}{r_s}}\left(dv^2 - du^2\right) + r^2 d\Omega^2 \qquad (1) \\ u^2 - v^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s}}$$

(1) Collapse of ~~a~~ spherical mass



(I) $-u < v < +u$, $u > 0$

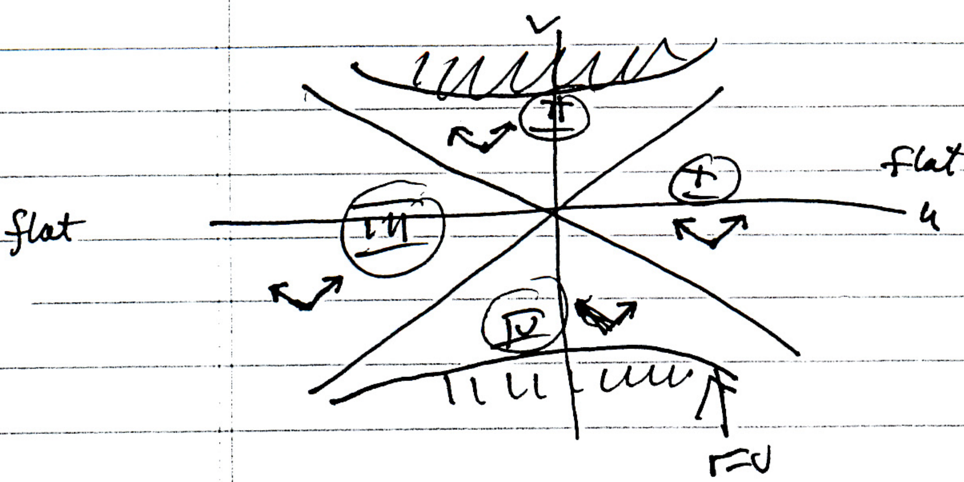
(II) $v > u$, $u > 0$

~~$v < u, u > 0$~~

$v > -u$, $u < 0$

Regions I, II are regions accessible to collapsed objects. I region at $r > r_s$, II, $r < r_s$; interior to BH event horizon. Particles moving from regions I to II reach $r = 0$

(2) Kruskal examined regions III, IV



- Regions III, IV inaccessible from I and II
- I & II causally disconnected
- Recall $r = 0$
  $u^2 - v^2 = -1$
  $v = \pm\sqrt{u^2+1}$
  — is white hole

- Region IV is similar to II but in reverse.

- Wormholes: I and III can connected by a wormhole at the origin.

Consider the geometry of the spacelike hypersurface $v=0$, which extends from $u=+\infty$ to $u=-\infty$. The line element for this hypersurface is

$$ds^2 = \frac{32(GM/c^2)^3 e^{-r/r_s}}{r} du^2 + r^2 d\Omega^2$$

Restrict to equatorial plane $\theta = \pi/2$ (spherical symmetry)

$$\therefore \boxed{ds^2 = \frac{32(GM/c^2)^3 e^{-r/r_s}}{r} du^2 + r^2 d\phi^2} \quad (2)$$

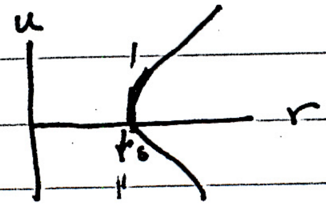
Interpret by letting 2-D surface with a line element $d\sigma^2$ given by last equation and embed it in 3-D Euclidean space (remember 2D sphere in 3D space)

In usual coordinates, make inverse transformation and you get back usual Schwarzschild line element

$$\boxed{d\sigma^2 = \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\phi^2} \quad (3)$$

Remember along surface $v=0$, as we decrease u from $+\infty$ to $-\infty$, r decreases to r_s

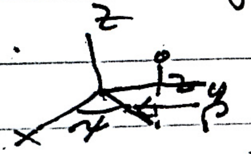
at $u=0$ and then r increases again as $u \to +\infty$



$$u = \pm\sqrt{\left(\frac{r}{r_s}-1\right)e^{r/r_s}}$$

The metric $d\sigma^2$ has rotational symmetry inherited from spherical symmetry of dS^2. This implies it might be possible to embed this slice in an axisymmetric surface in 3-D flatspace.

<u>Flat space</u>: Using cylindrical coordinates ρ, ψ, z



$$\boxed{d\Sigma^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \quad (4)}$$ (flat space metric)

surface in flat space given by height above $z=0$ plane at each point: $z = z(r,\phi)$. But we need a connection between coordinates ρ, ψ in (4) with r, ϕ in (3)

$$z = z(r,\phi)\ ;\ \rho = \rho(r,\phi)\ ;\ \psi = \psi(r,\phi)$$

Since surface is axisymmetric:

[illegible] ; $z = z(r),\ \rho = \rho(r),\ \psi = \phi$ [illegible]

~~On that case~~ $\cancel{d\Sigma^2 = dr^2 + r^2 d\psi^2 + dz^2}$

$\cancel{d\Sigma^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2}$

Demanding $d\Sigma^2 = d\sigma^2 \Rightarrow$

$$d\rho^2 + \rho^2 d\phi^2 + dz^2 = \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\phi^2 \Rightarrow \rho = r$$

Therefore we have:

$$d\sigma^2 = dr^2 + r^2 d\phi^2 + dz^2$$

$$= \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2 = \left[1 - \frac{r_s}{r}\right]^{-1} dr^2 + r^2 d\phi^2$$

As a result: $1 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1 - \frac{r_s}{r}}$

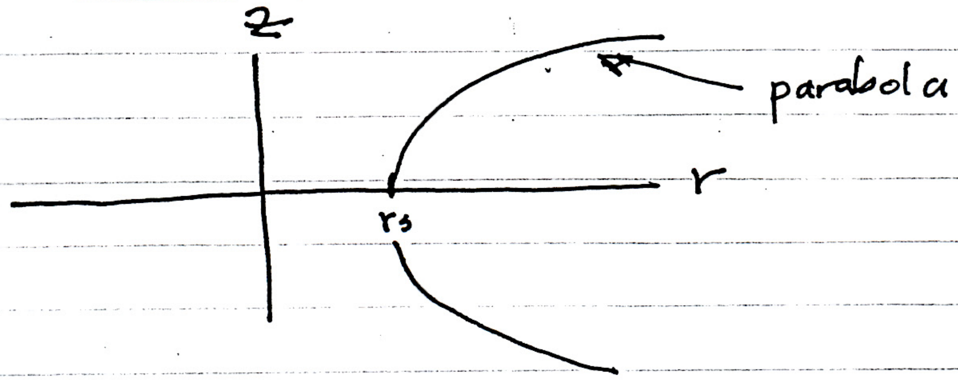
or $\left(\frac{dz}{d\sigma}\right)^2 = \frac{1}{1 - \frac{r_s}{r}} - 1 = \frac{1 - \left(1 - \frac{r_s}{r}\right)}{1 - \frac{r_s}{r}}$.

$$\left(\frac{dz}{dr}\right)^2 = \frac{r_s/r}{1 - r_s/r} = \frac{1}{\frac{r}{r_s} - 1}$$

$$dz = \frac{dr}{\sqrt{\frac{r}{r_s} - 1}}$$

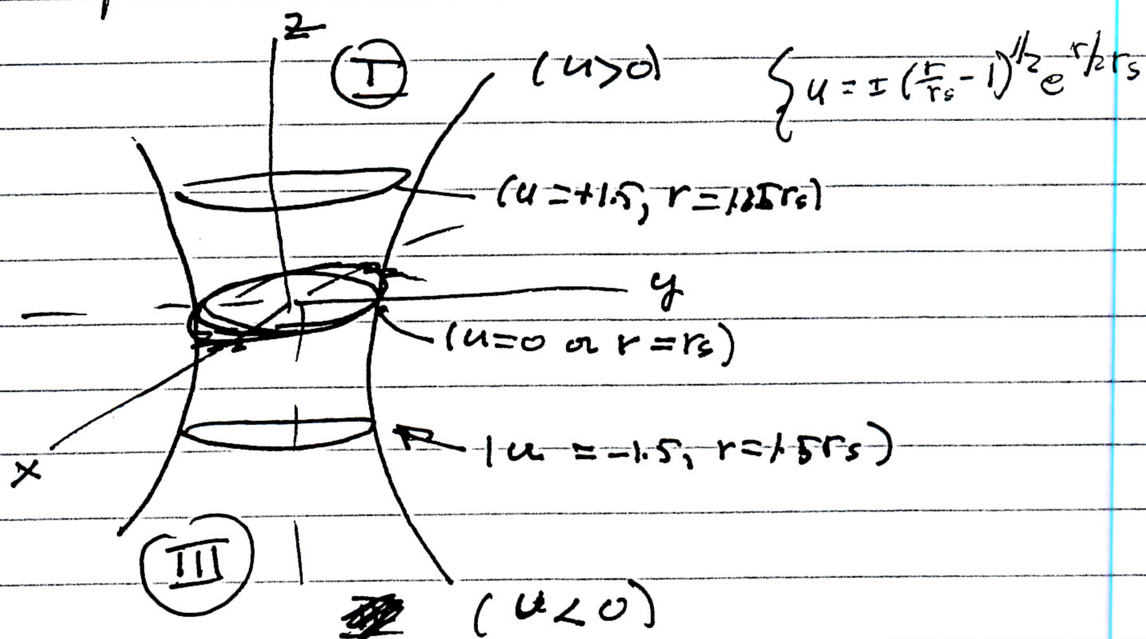
$$z(r) = \int_0^r \frac{dr'}{\sqrt{\frac{r'}{r_s} - 1}}$$

$$\boxed{z(r) = \sqrt{4 r_s (r - r_s)}}$$



Rotate parabola about z axis and find

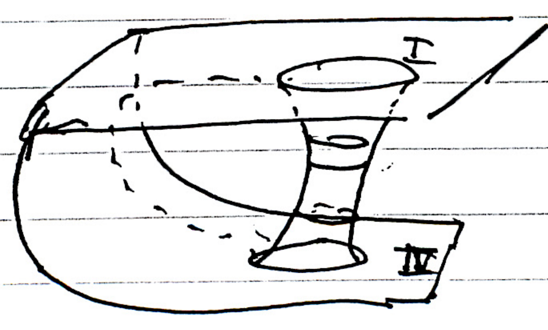
z(r) is a paraboloid



We have a throat or "wormhole" that physically connects regions I and III

Sometimes called: Einstein / Rosen bridge.

In principle one can connect two distant regions like so:



Universe is 2D sheet. ant living on surface oblivious to global geometry. Distance along dotted line much longer than shorter connection through wormhole. Possibility of shorter distance space travel to distant universe!

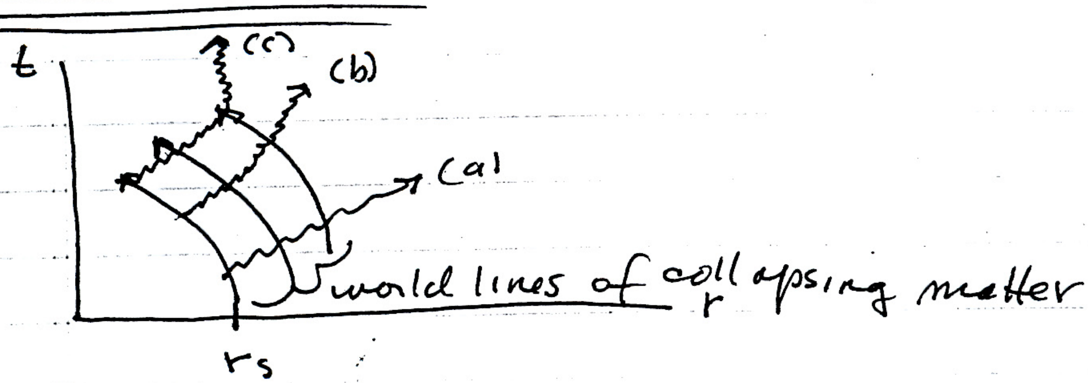
Realistic Collapse

Geometry outside collapsing spherical star is Schwarzschild. But inside it may be very different.

More General Black Holes

Formation. How does horizon appear when star collapses within its Schwarzschild radius? Stated differently: How does horizon grow from $r_H = 0$ to $r_H = 2GM/c^2 = r_s$?

Schematic History



(a) Photons get out early (Most of star at $r \gg r_s$)
(b) Photons experience some delay (surface getting near r_s)
(c) Photon is "marginal" one that gets trapped on Schwarzschild horizon. Distant observer sees horizon grow from $r=0$ to $r = 2GM/c^2$ by viewing trajectory of (c).

General Results

Nearly Spherical Collapse

All non-spherical parts of mass distribution are "radiated" away by gravitational radiation. If there is no spin, one winds up with spherical Schwarzschild solution.

Stationary Horizons

After every thing settles down, horizon becomes stationary. As a result BH is characterized by

M : mass

Q : net charge

J : angular momentum

In reality only M and J are relevant since charge neutrality is quickly established.

Hawking Area Theorem

Any Physical process involving the horizon, one finds that the area of the horizon can only increase with time.

Spinning Kerr Black Holes

Metric

$$-ds^2 = \left(1 - \frac{2\left(\frac{GM}{c^2}\right)r}{\Sigma}\right)(c\,dt)^2 + \frac{4a\,(GM/c^2)\,r\sin^2\theta\, c\,dt\,d\phi}{\Sigma'} - \frac{\Sigma'}{\Delta}\,dr^2 - \Sigma\, d\theta^2$$

$$- \left(r^2 + a^2 + \frac{2\left(\frac{GM}{c^2}\right) r a^2 \sin^2\theta}{\Sigma'}\right)\sin^2\theta\, d\phi^2 \qquad (1)$$

where $a \equiv \frac{J}{Mc}$; $\Delta \equiv r^2 - 2\left(\frac{GM}{c^2}\right)r + a^2$; $\Sigma' = r^2 + a^2\cos^2\theta$

Comments:

- Metric is stationary (independent of t)
- " " axisymmetric (independent of ϕ)
- θ, ϕ, r are usual spherical coordinates
- Limit $a \to 0$
  - (i) cross term $(dt\,d\phi)$ coefficient vanishes
  - (ii) $\Delta \Rightarrow r^2\left(1 - \frac{2GM/c^2}{r}\right)$

    $\Sigma' \to r^2$

In that case: $-ds^2 = \left(1 - \frac{(2GM/c^2)r}{r^2}\right)(c\,dt)^2 - \frac{r^2\, dr^2}{r^2\left(1 - \frac{2GM/c^2}{r}\right)}$

$$- r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

$$-ds^2 = \left(1 - \frac{r_s}{r}\right)(c\,dt)^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} - r^2 d\Omega^2 \quad : \text{Schwarzschild}$$

Singularities

Looking at eq. (1) we see that metric is singular when:

(A) $\Sigma = 0$ and when (B) $\Delta = 0$

2/26-02

(A) Let's look at Σ_1 condition first.

From eq. (1) this happens when

$$r^2 + a^2\cos^2\theta = 0$$

i.e., when $r=0$, $\theta = \pi/2$

this is a real singularity analogous to $r=0$ singularity of Schwarzschild spacetime. True because curvature tensor R_{abcd} blows up at these points; i.e., gauge invariant tidal forces

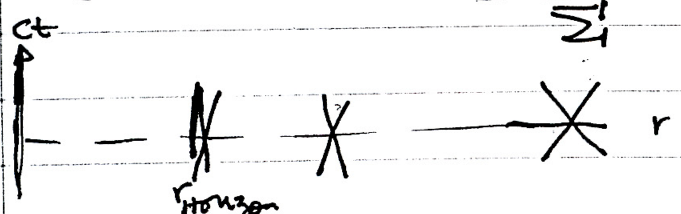
(B) Next look at Δ condition

It turns out that $\Delta = 0$ is condition for horizon!

Recall radial light rays ($ds^2=0$, $d\theta = d\phi = 0$)

$$\left(1 - \frac{r_s r}{\Sigma}\right)(c\,dt)^2 - \frac{\Sigma_1}{\Delta}\,dr^2 = 0 \qquad r_s = 2GM/c^2$$

$$\left(\frac{c\,dt}{dr}\right)^2 = \frac{\Sigma_1}{\Delta} \times \frac{1}{1 - \frac{r_s r}{\Sigma_1}}$$



Horizon located where $c\,dt/dr \to \infty$ (photon trapped at $r = \text{const} = r_H$)

But this occurs where $\Delta \to 0$

From eq. (1) we have:

$$\Delta = r^2 - r_s r + a^2 = 0 \qquad \left\{ r_s \equiv \frac{2GM}{c^2} \right\}$$

$$r_\pm = \frac{r_s \pm \sqrt{r_s^2 - 4a^2}}{2}$$

$$\boxed{r_\pm = \frac{r_s}{2} \pm \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}} \quad (2)$$

Comments

- Unlike singularity at $r=0$, these singularities are coordinate singularities, since curvature tensor does not diverge at $r_\pm$. Not surprise since $r_\pm \to r_s$ in the limit $a \to 0$

- For BH to exist $a^2 < \left(\frac{r_s}{2}\right)^2$ (For $r_\pm$ to be real)

  Physically this means

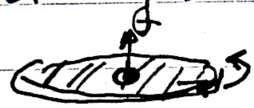
$$\frac{J}{Mc} < \frac{GM}{c^2}$$

$$\text{or } J < \frac{GM^2}{c} = \frac{GM}{c^2}(Mc)$$

is maximum angular momentum allowed for BH and for horizons to exist. BHs with $J = \frac{GM^2}{c}$ are extreme black holes in which $r_+ = r_- = GM/c^2$ (compare Schwarzschild where horizon is at $r_s = \frac{2GM}{c^2}$)

Does this maximum value of J make sense?

It may be that extreme Kerr BHs develop when matter spirals in toward Kerr BH forming an accretion disk rotating in same sense as BH



As matter from disk falls in, it carries angular momentum with it, which increases J of BH. However, radiation emitted by infalling matter carries away J which may limit $J < \frac{GM}{c^2}(Mc) \simeq \frac{r_s}{2} \cdot Mc$

But, note condition $a^2 > \left(\frac{r_s}{2}\right)^2$

$J > \frac{GM}{c^2}(Mc)$: what happens in this case?

2-26-04

Kerr metric well behaved everywhere except at $r=0$, $\theta=\pi/2$ where physical singularity exists. (Note $r=0$ corresponds a disk of coordinate radius $a = J/Mc$ in plane $\theta=\pi/2$). ~~Som~~

~~If~~ Since $r_\pm$ are not real, horizons would not exist in this case and in principle one could peer into "naked singularity", but various theorems such as the Penrose "Cosmic ~~Censor~~ Censorship ~~theorem~~ conjecture" ~~rule this out~~. argue against this. We shall see that Hawking area theorem also argues against this possibility.

Engosphere ⟶

Recap: Kerr Metric

Kerr Metric for spinning black hole

$$-ds^2 = \left(1 - \frac{\left(\frac{2GM}{c^2}\right) r}{\Sigma}\right)(c\,dt)^2 + \frac{4a\left(\frac{GM}{c^2}\right) r \sin^2\theta \, c\,dt\,d\phi}{\Sigma} - \frac{\Sigma}{\Delta} dr^2$$

$$-\Sigma\, d\theta^2 - \left(r^2 + a^2 + \frac{2\left(\frac{GM}{c^2}\right) r\, a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2$$

where $a = J/Mc$, $\Delta = r^2 - 2\left(\frac{GM}{c^2}\right) r + a^2$; $\Sigma = r^2 + a^2\cos^2\theta$

Singularities: where $ds^2 \to \infty$

(A) $\{\Sigma = 0\} \Rightarrow r = 0, \theta = \pi/2$.

True physical singularity like $r=0$ in Schwarzschild, but because of weird geometry, this corresponds to a ring with radius $= a$ in equatorial plane. (these are not usual polar coordinates as in Schwarzschild)

(B) $\{\Delta = 0\}$ Coordinate singularity like Schwarzschild event horizon at $r = 2GM/c^2$. As in that case light-like geodesic moves in time-like direction; i.e. $c\,dt/dr \to \infty$

Two horizons: $r_\pm = \frac{r_s}{2} \pm \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}$

where $r_s = 2GM/c^2$

Surfaces $r = r_\pm$ are not spheres but axially symmetric. If we embed these 2D surfaces in 3D Euclidean space I get axisymmetric ellipsoids flattened along rotation axis.

r_+ outer horizon does not emit light rays.

(C) Extreme Kerr BH's

$r_\pm$ ~~poss~~ exist only if $a^2 < \left(\frac{GM}{c^2}\right)^2$

$$\frac{J}{Mc} < \frac{GM}{c^2} \Rightarrow J < J_{Limit} = \left(\frac{GM}{c^2}\right) Mc$$

(D) Now lets discuss limits on stationary orbits

$\longrightarrow$

Ergosphere:

Effect of angular momentum and rotation on spacetime geometry.

Consider observers who remain stationary wrst to $r=\infty$; i.e., no time-dependent change of their coordinates. Rockets on!.

But, even with rockets on, there is a limit to how close stationary observer can get to BH and remain stationary.

→ 4 velocity of timelike observer: $u^a = \frac{dx^a}{d\tau}$

time component: $u^t \equiv c\,dt/d\tau$

radial space component: $u^r = dr/d\tau = 0$ (stationary)

$\therefore$ $ds^2 = g_{ab}dx^a dx^b \Rightarrow \left(\frac{ds}{d\tau}\right)^2 = g_{tt}\left(\frac{c\,dt}{d\tau}\right)^2$ (stationary)

Recall definition of proper time: $ds^2 = -c^2 d\tau^2$

Therefore: $g_{tt}\left(\frac{c\,dt}{d\tau}\right)^2 = -c^2$

Plugging in Kerr metric value for g_{tt},

$$-\left(1 - \frac{\left(\frac{2GM}{c^2}\right)r}{\Sigma}\right)\left(\frac{c\,dt}{d\tau}\right)^2 = -c^2$$

$$\text{or} \quad \left(1 - \frac{2\left(\frac{GM}{c^2}\right)r}{\Sigma}\right)\left(\frac{dt}{d\tau}\right)^2 = 1$$

at points where this condition is satisfied, and hence $u^t = c\,dt/d\tau$

$$(u^t)^2 = \frac{c^2}{1 - \frac{2\left(\frac{GM}{c^2}\right)r}{r^2+a^2\cos^2\theta}} = \frac{c^2(r^2+a^2\cos^2\theta)}{r^2+a^2\cos\theta-2\left(\frac{GM}{c^2}\right)r}$$

But as r decreases it cannot be satisfied at all points. Specifically (r^2 term decrease faster than r; so denominator heads $\to 0$ and < 0)

(a) Denominator vanishes ($u^t \to \infty$) when

$$1 - \frac{2\left(\frac{GM}{c^2}\right)r}{\Sigma} = 0$$

$$\text{or when} \quad \Sigma - 2\left(\frac{GM}{c^2}\right)r = 0$$

$$r^2 + a^2\cos^2\theta - 2\left(\frac{GM}{c^2}\right)r = 0$$

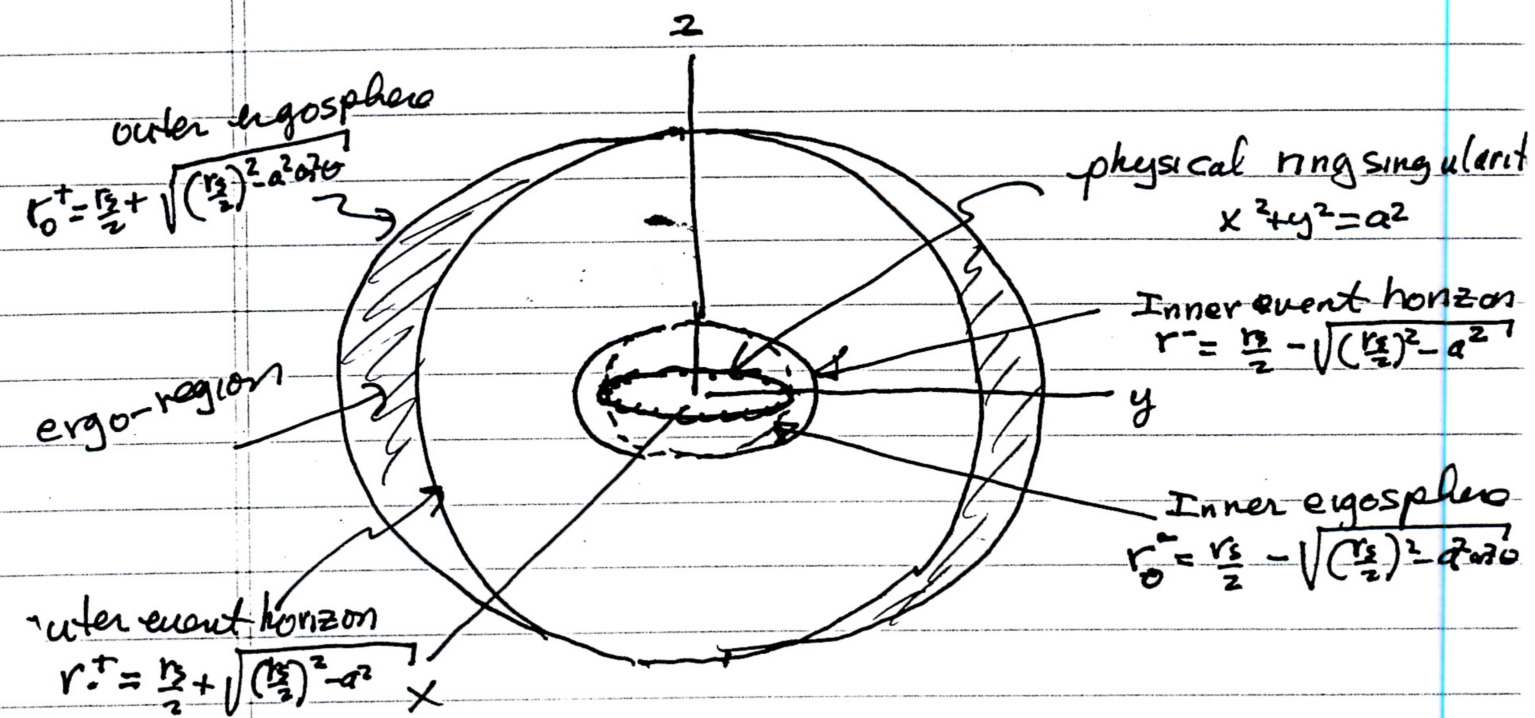
Solve quadratic

$$r_0^{\pm} = \frac{2\left(\frac{GM}{c^2}\right) \pm \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4a^2\cos^2\theta}}{2}$$

Thus stationary observer impossible on surface

$$\boxed{r_0^{\pm} = \left(\frac{GM}{c^2}\right) \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta}}$$

Surface is also an axisymetric



Ergo-Region

- No stationary orbits here since u^t is imaginary

- But particles can enter & escape from this region since it is outside horizon

- particle entering ergosphere on pure radial orbit develops tangential velocity: it is dragged around by gravitational field of Kerr metric

→ 2

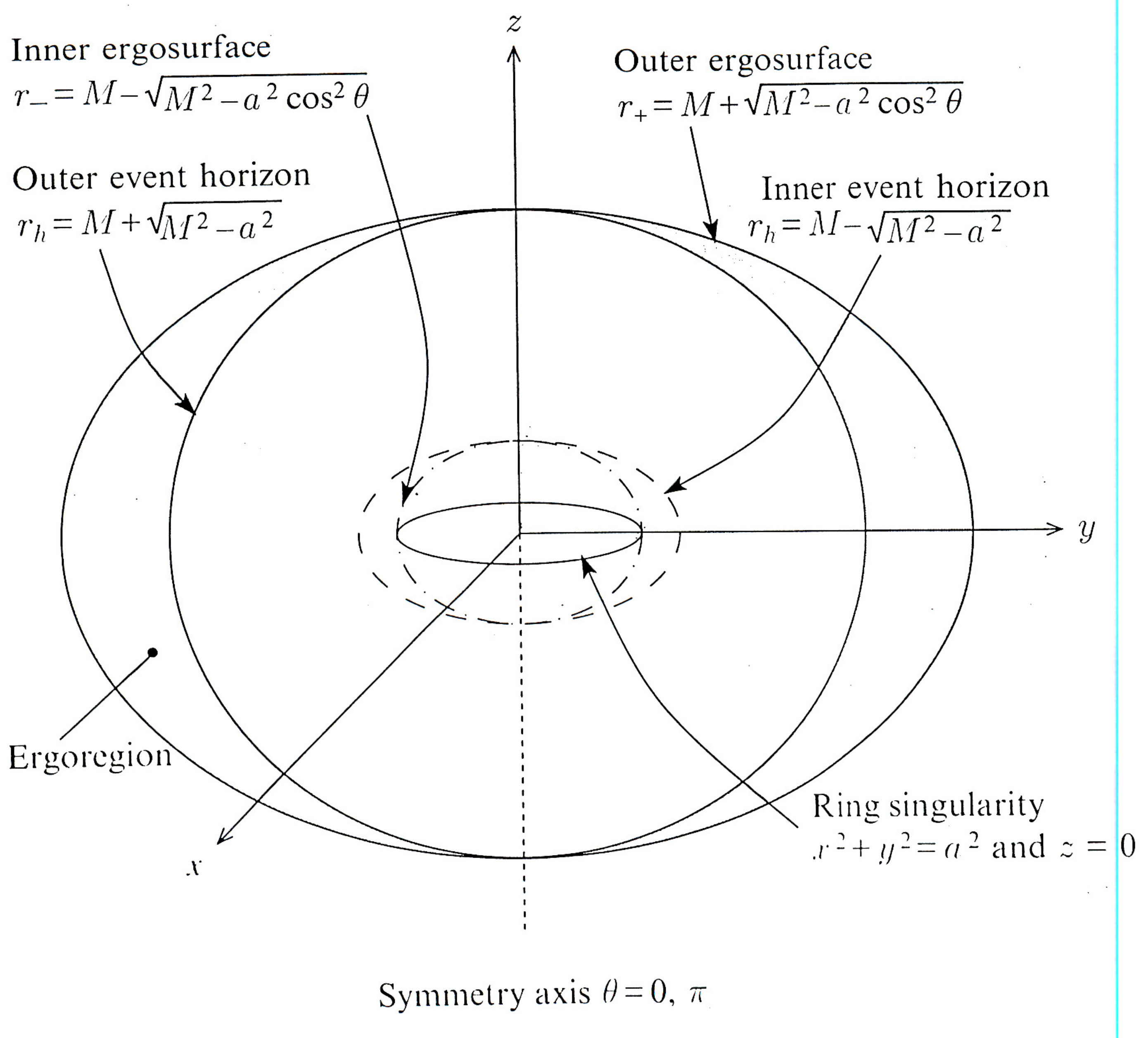


Fig. 8.9. Schematic picture showing the geometrical structure of the Kerr spacetime.

The region between theses two surfaces is called the *ergosphere* and we shall now describe the interesting new phenomena the existence of ergosphere leads to. Figure 8.10 shows the Penrose–Carter diagram for the Kerr metric.

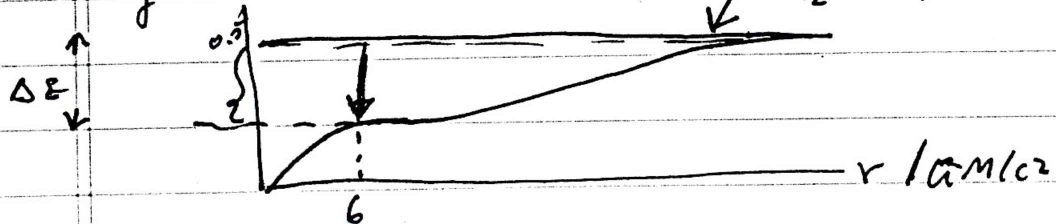
Exercise 8.8
Closed timelike curves in the Kerr metric Prove that the Kerr metric, when analytically

2-26-09

Energy Extraction from BH:

How much binding energy per unit mass can we extract from a particle that comes in from $r=\infty$ and settles into the last stable circular orbit?

Schwarz. We solved this problem for Schwarzschild Geometry.



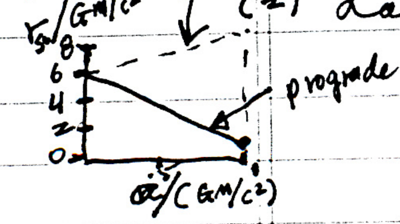
Energy released as particle settles in at $r=6GM/c^2$!

$$\frac{\Delta E}{mc^2} = \frac{mc^2 - \mathcal{E}}{mc^2} = 1 - \mathcal{E} = 1 - \sqrt{\frac{8}{9}} = 1 - .942 = .05$$

Kerr. Much more complicated (Good term paper).

(1) Solve Geodesic Equations for timelike orbits

(2) Last stable circular orbit:



- $a = GM/c^2$: $r_{so} = GM/c^2$: direct (prograde); $r_{so} = 9GM/c^2$: retrograde
- $a = 0$: $r = 6GM/c^2$

$\frac{\mathcal{E}}{mc^2}$ decreases from $\sqrt{\frac{8}{9}}$ ($a=0$) to $\sqrt{\frac{1}{3}}$ ($a=GM/c^2$) direct (prograde)

" " $\sqrt{\frac{25}{27}}$ $a=\frac{GM}{c^2}$: retrograde

$$\therefore \frac{\Delta E}{mc^2} = 1 - \frac{1}{\sqrt{3}} = 0.423 \quad \text{for } a = GM/c^2 \text{: direct}$$

$$= 0.057 \quad a = 0$$

2-26-10

This is another important reason why BHs attract attention as quasar energy sources.

Area Theorem and BH Evaporation

In 1970's Hawking proved a remarkable theorem about BHs:

- In any interaction the surface area of the horizon never decreases. For multiple BHs, sum of the areas never decreases

(1) Horizon area for Kerr BH

what we mean by area is 2D surface across which $t = \text{const}$, $r = r_+ = \text{const}$, i.e., $dt = dr = 0$

thus spatial metric from eq. (1) becomes

$$ds^2 = \Sigma_+ d\theta^2 + \left\{ (r^2+a^2) + \frac{2\frac{GM}{c^2} r_+ a^2 \sin^2\theta}{\Sigma_+} \right\} \sin^2\theta d\phi^2$$

$\Sigma_+ = \Sigma(r_+) = r_+^2 + a^2\cos^2\theta$

$$= \Sigma_+ d\theta^2 + \left\{ \frac{(r_+^2+a^2)\Sigma_+ + \frac{2GM}{c^2} r_+ a^2 \sin^2\theta}{\Sigma_+} \right\} \sin^2\theta d\phi^2$$

But along horizon $\Delta = 0$ where $\Delta = r^2 - 2\left(\frac{GM}{c^2}\right) r + a^2 = 0$

or $r_+^2 + a^2 = 2\left(\frac{GM}{c^2}\right) r_+$

$$ds^2 = \Sigma_+ d\theta^2 + \left\{ \frac{2\left(\frac{GM}{c^2}\right) r_+ \left[\Sigma_+ + a^2\sin^2\theta \right]}{\Sigma_+} \right\} \sin^2\theta d\phi^2$$

But $\Sigma_+ = r_+^2 + a^2\cos^2\theta$

$2\left(\frac{GM}{c^2}\right) r_+$

$$\therefore ds^2 = \Sigma_+ d\theta^2 + \left\{ \frac{2\left(\frac{GM}{c^2}\right) r_+ \cdot (r_+^2 + a^2)}{\Sigma_+} \right\} \sin^2\theta d\phi^2$$

2-26-11

$$\therefore\ ds^2 = \Sigma_{1+}\, d\theta^2 + \frac{4\left[\left(\frac{GM}{c^2}\right) r_+\right]^2}{\Sigma_{1+}} \sin^2\theta\, d\phi^2$$

But $A(r_+) = \iint \sqrt{\det g}\; d\theta\, d\phi$ (theorem for curved spaces)

$$g_{ab} = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & \frac{4\left[\frac{GM}{c^2} r_+\right]^2 \sin^2\theta}{\Sigma_+} \end{pmatrix} \Rightarrow \det g = 4\left[\frac{GM}{c^2} r_+\right]^2 \sin^2\theta$$

Since $\sqrt{\det g} = 2\frac{GM}{c^2} r_+ \sin\theta$,

$$\therefore\ A(r_+) = 2\frac{GM}{c^2} r_+ \iint d\phi \sin\theta\, d\theta = 2\frac{GM}{c^2} r_+ \cdot 4\pi$$

$$\boxed{A(r_+) = 8\pi \frac{GM}{c^2} r_+}$$

$$\boxed{A = 8\pi \frac{GM}{c^2}\left[\frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}\right]}$$

We can now use Hawking's area theorem to show that one cannot make a "naked singularity" in which $a > GM/c^2$

Suppose we increase M of BH by δM.

then Let $\tilde{m} = GM/c^2$

$$\delta A = 8\pi\, \delta\left[\tilde{m}^2 + \tilde{m}\sqrt{\tilde{m}^2 - a^2}\right]$$

$$= 8\pi\left[2\tilde{m}\,\delta\tilde{m} + \delta\tilde{m}\sqrt{\tilde{m}^2 - a^2} + \frac{\tilde{m}}{2}\left(\tilde{m}^2 - a^2\right)^{-1/2} 2\tilde{m}\right]$$

Let's look first at simple case of a Schwarzschild hole, $a=0$. In that case

$$\boxed{A = 8\pi \frac{GM}{c^2} \times 2\frac{GM}{c^2} = 16\pi \left(\frac{GM}{c^2}\right)^2 \quad , a=0}$$

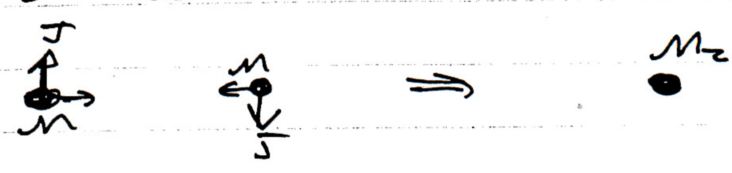
- Surface area of non-rotating BH is proportional to square of mass.

- Classically, mass-energy can only fall into BH, never falls out! So easy to see why $A \uparrow$)

- So area theorem just says that since area positive function of mass. Hawking's area theorem just generalizes this to rotating holes and replaces mass with horizon area.

Applications

(1) 2 Kerr holes of equal mass M but opposite angular momentum collide to produce a Schwarzschild hole of mass M_2 and net $a=0$



Let's compare M_2 with $2M$!

Area theorem says:

$$A(M_2, J=0) > A(M, J) + A(M, -J)$$

Let $\tilde{m} \equiv GM/c^2$ each has same area

$$A = 8\pi\tilde{m}\left[\tilde{m} + \sqrt{\tilde{m}^2 - a^2}\right]$$

$$8\pi\tilde{M}_2(\tilde{M}_2 + \tilde{M}_2) > 8\pi\tilde{m}\left[\tilde{m} + \sqrt{\tilde{m}^2 - a^2}\right] \times 2$$

$$\tilde{M}_2(\cancel{2}\tilde{M}_2) > \tilde{m}\left(\tilde{m} + \sqrt{\tilde{m}^2 - a^2}\right)\cancel{2}$$

$$\tilde{M}_2^2 > \tilde{m}^2 + \tilde{m}\sqrt{\tilde{m}^2 - a^2} = \tilde{m}^2 + \tilde{m}^2\sqrt{1 - \left(\tfrac{a}{\tilde{m}}\right)^2}$$

$$\boxed{\tilde{M}_2^2 > m^2\left[1 + \sqrt{1 - \left(\tfrac{a}{\tilde{m}}\right)^2}\right]}$$

Extreme case: $a = \tilde{m}$

$$\Rightarrow \tilde{M}_2^2 > m^2$$

$$M_2 > m \quad \text{or}$$

Mass loss: $$\frac{\Delta m}{2m} = \frac{\overset{\text{initial}}{2m} - \overset{\text{final}}{M_2}}{2m} = 1 - \frac{M_2}{2m}$$

So if $M_2 \geq m$, the

$$\boxed{\frac{\Delta m}{2m} \leq 0.5}$$

So in principle 50% of mass can be lost in this process. Where does it go? Gravitational radiation! This is most efficient case since $a < \tilde{m}$ leads to higher M_2

(2) Suppose the colliding holes are Schwarzschild holes; i.e., $a = 0$

In that case

$$M_2^2 > M^2\left[1 + \sqrt{1-0}\right] = 2M^2$$

$$M_2 > \sqrt{2}\,M$$

In that case $\dfrac{\Delta M}{2M} = 1 - \dfrac{M_2}{2M} < 1 - \dfrac{\sqrt{2}M}{2M}$

$$\frac{\Delta M}{M} < 1 - 0.714 < 0.29$$

(3) Impossibility of Naked Singularity

Hawking's theorem also ~~says~~ implies that one cannot make a "naked singularity" in which $a > GM/c^2$ (resulting in imaginary horizons that do not exist).

Recall: $A = 8\pi \tilde{M}\left[\tilde{M} + (\tilde{M}^2 - a^2)^{1/2}\right]$

Suppose BH mass changes by $\delta\tilde{M}$

BH J/Mc changes by δa

Since $A = A(\tilde{M}, a)$, we have

$$\delta A = \frac{\partial A}{\partial \tilde{M}}\,\delta\tilde{M} + \frac{\partial A}{\partial a}\,\delta a$$

Ignoring 8π factor: $A = \tilde{M}^2 + \tilde{M}(\tilde{M}^2 - a^2)^{1/2}$

$$\frac{\partial A}{\partial \tilde{M}} = 2\tilde{M} + (\tilde{M}^2 - a^2)^{1/2} \quad \frac{1}{2}(\tilde{M}^2 - a^2)^{-1/2}(2\tilde{M})\times\tilde{M}$$

$$= 2\tilde{M} + (\tilde{M}^2 - a^2)^{1/2} + \frac{\tilde{M}^2}{(\tilde{M}^2 - a^2)^{1/2}}$$

$$\frac{\partial A}{\partial \tilde{M}} = \frac{2\tilde{M}(\tilde{M}^2 - a^2)^{1/2} + \tilde{M}^2 - a^2 + \tilde{M}^2}{(\tilde{M}^2 - a^2)^{1/2}}$$

$$\boxed{\frac{\partial A}{\partial \tilde{M}} = \frac{2\tilde{M}(\tilde{M}^2 - a^2)^{1/2} + 2\tilde{M}^2 - a^2}{(\tilde{M}^2 - a^2)^{1/2}}}$$

$$\boxed{\frac{\partial A}{\partial a} = \frac{1}{2}(\tilde{M}^2 - a^2)^{-1/2}(-2a)\,\tilde{M} = \frac{-\tilde{M}a}{(\tilde{M}^2 - a^2)^{1/2}}}$$

Therefore
$$\delta A = \frac{\left[2\tilde{M}(\tilde{M}^2 - a^2)^{1/2} + 2\tilde{M}^2 - a^2\right]\delta M - \left[\tilde{M}a\right]\delta a}{(\tilde{M}^2 - a^2)^{1/2}}$$

Condition $\delta A > 0$ implies

$$\boxed{\left[2\tilde{M}(\tilde{M}^2 - a^2)^{1/2} + 2\tilde{M}^2 - a^2\right]\delta\tilde{M} > \left[\tilde{M}a\right]\delta a}$$

Extreme case of $a \to \hat{m}$ (since this is where naked singularity might exist)

Last equation implies:

$$\left(0 + 2\hat{M}^2 - \hat{m}^2\right)\delta\hat{m} > \hat{m}\, a\,\delta a$$

$$\hat{m}\,\delta\hat{m} > a\,\delta a$$

$$\boxed{\delta(m^2) > \delta(a^2)}$$

So: dumping mass and angular momentum onto a Kerr hole increases its mass faster than its a. As a result if we start out with real horizon in which $\hat{m}^2 > a^2$, we stay with that inequality. As a result, area of horizon remains real & does not disappear –

Black Hole Thermodynamics

Hawkings area theorem sounds alot like second law of thermodynamics in which entropy of a closed system always increases.

Beckenstein (1972) proposed that surface area of a BH's event horizon is proportional to its entropy; i.e., that a BH is physically a black body with all the radiation properties of a black body. But to have physical meaning one must