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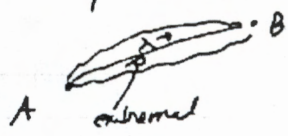
WEEK 5: GEODESICS IN SCHWARZSCHILD Spacetime

Recap: Previous week I used calculus of variations; which is a methodology for finding geodesics; i.e., extremal paths between 2 spacetime events A, B.



λ: parameter measuring distance along path.

These are paths followed by particles in free fall without external non-gravitational forces.

Let $S = \int_{A\ \text{path}}^{B} \mathcal{L}(\underset{\sim}{x}, \dot{\underset{\sim}{x}}, \lambda)\, d\lambda$

Geodesics equations followed from variational principle, that $\delta S = 0$; i.e., geodesics are extremal curves. In this formalism

(a) λ is "affine" parameter that varies along path

(b) 4 vectors $\begin{cases} \underset{\sim}{x} = (x, y, z, t) & \text{left this out last time} \\ \dot{\underset{\sim}{x}} = \left(\frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda}, \frac{dt}{d\lambda}\right) \end{cases}$

$x(\lambda), y(\lambda), z(\lambda), t(\lambda)$ are solutions to Euler-Lagrange eqs.

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

I solved these in 1 case

① Flat space : $ds^2 = \sqrt{dx^2 + dy^2 + dz^2}$

$$\mathcal{L} = \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2}$$

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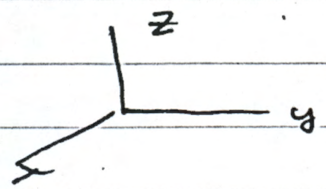
We saw that these are space-like geodesics in flat Euclidean Space:



$y = m_1 x + b_1$; $z = m_2 x + b_2$; $z = m_3 y + b$

② Mechanics

Here we do <u>not</u> find extremal values of the interval. Rather we find extremal values of the action

$$S = \int \mathcal{L}\, d\lambda$$

where Lagrangean: $\mathcal{L} = K - V = \frac{1}{2} m \dot{x}^2 - V(x)$

In classical mechanics $\lambda = t$

<u>Lagrange equations</u>

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V}{\partial x} \quad ; \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Therefore: $-\frac{\partial V}{\partial x} - \frac{d}{dt}(m\dot{x}) = 0$

or $\boxed{m\ddot{x} = -\frac{\partial V}{\partial x}}$ Newton's 2^{nd} Law

③ Special Relativity

$$-ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

In this case $\mathcal{L} = \sqrt{-\left(\frac{ds}{d\lambda}\right)^2}$

Use proper time $\tau = \lambda$ $\quad d\tau = \sqrt{-ds^2}$

$$-\left(\frac{ds}{d\lambda}\right)^2 = c^2\frac{dt}{d\lambda^2} - \left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dy}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2$$

$$\sqrt{-\left(\frac{ds}{d\lambda}\right)^2} = c\sqrt{\dot{t}^2 - \frac{1}{c^2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)} = \mathcal{L}(x,y,z;\dot{x},\dot{y},\dot{z};\lambda$$

x: $\quad \frac{\partial\mathcal{L}}{\partial x} = 0 \; ; \; \frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{c\left(-\frac{2\dot{x}}{c^2}\right)}{2\sqrt{}} = \frac{-\dot{x}/c}{\sqrt{}}$

<u>Consider time-like geodesics:</u>

$\lambda = \tau$: In this case $\left(-\frac{ds}{d\lambda}\right)^2 = -\left(\frac{ds}{d\tau}\right)^2 = c^2$

therefore $\sqrt{-\left(\frac{ds}{d\lambda}\right)^2} = c = c\sqrt{\dot{t}^2 - \frac{1}{c^2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)}$

$$\Rightarrow \sqrt{} = 1$$

So $\quad \frac{\partial\mathcal{L}}{\partial\dot{x}} = -\dot{x}/c \; ; \; \frac{\partial\mathcal{L}}{\partial\dot{y}} = -\dot{y}/c \; ; \; \frac{\partial\mathcal{L}}{\partial\dot{z}} = -\dot{z}/c$

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = c\dot{t}$$

Since $\frac{\partial\mathcal{L}}{\partial x} = 0$, $\frac{\partial\mathcal{L}}{\partial t} = 0$

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) ; \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0$$

$$\frac{d}{d\tau}\left(-\frac{\dot{x}}{c}\right) = 0 \Rightarrow x = c_1\tau + x_0$$

$$\frac{d}{d\tau}(c\dot{t}) = 0 \Rightarrow t = c_2\tau + t_0$$

$$(\text{set } t_0 = 0) \Rightarrow \boxed{t = c_2\tau}$$

Therefore: $x = c_1\left(\frac{t}{c_2}\right) + x_0$

$$\frac{dx}{dt} = v_x = \frac{c_1}{c_2} \quad (3 \text{ velocity})$$

Now ignore $\dot{y}, \dot{z}$

Recall $\sqrt{\dot{t}^2 - \frac{1}{c^2}\dot{x}^2} = 1$

$$\therefore \sqrt{c_2^2 - \frac{c_1^2}{c^2}} = \sqrt{c_2^2 - \frac{v_x^2 c_2^2}{c^2}} = c_2\sqrt{1 - \left(\frac{v_x}{c}\right)^2} = 1$$

Therefore $\boxed{c_2 = \frac{1}{\sqrt{1 - \left(\frac{v_x}{c}\right)^2}} = \gamma}$

But since $t = c_2\tau \Rightarrow \boxed{t = \gamma\tau}$.

$$x = v_x t + x_0 \quad \text{or} \quad x = (v_x\gamma)\tau + x_0$$

Eq. of motion of special relativity

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$\mathcal{L} = c$ valid for time-like geodesics

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Clearly: $\frac{\partial \mathcal{L}}{\partial y} = -\frac{\dot{y}}{c}$; $\frac{\partial \mathcal{L}}{\partial z} = -\frac{\dot{z}}{c}$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{c\,(2\dot{t})}{2\sqrt{\quad}} = c\dot{t}$$

Since $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial t} = 0$, we have

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \quad \text{and} \quad \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0$$

As a result:

$$\dot{x} = c_1 \quad ; \quad \dot{t} = c_2$$

Solve: $x = c_1\tau + x_0$; $t = c_2\tau$ (synchronize at $t=\tau=0$)

or $x = \left(\frac{c_1}{c_2}\right)t + x_0 \Rightarrow v_x = c_1/c_2$ (3-velocity)

Recall: $\sqrt{\dot{t}^2 - \frac{\dot{x}^2}{c^2}} = 1$ (Ignore y, z ..)

$$\therefore \sqrt{c_2^2 - \frac{c_1^2}{c^2}} = \sqrt{c_2^2 - \frac{c_2^2 v_x^2}{c^2}} = c_2\sqrt{1 - \frac{v_x^2}{c^2}} = 1$$

$$\therefore \quad c_2 = \frac{1}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \gamma$$

But since $t = c_2\tau \Rightarrow t = \gamma\tau$ (Time dilation)

$$\therefore \quad x = v_x t + x_0 ; \quad \tau = t/\gamma$$

These are geodesics in SR

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Energy & Momentum in SR

Canonical Coordinates and Conserved Quantities

Define canonical 4 momentum: $p_\mu \stackrel{d}{=} \dfrac{\partial \mathcal{L}}{\partial \dot{x}_\mu}$

where $\mu = 0, 1, 2, 3$ $\quad (t, x, y, z)$

p_μ conserved when $\dfrac{\partial \mathcal{L}}{\partial x_\mu} = 0$

why? Because $\dfrac{\partial \mathcal{L}}{\partial x_\mu} - \dfrac{d}{d\lambda}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_\mu}\right) = 0 \Rightarrow$

$$\frac{\partial \mathcal{L}}{\partial x_\mu} - \frac{d}{d\lambda}(p_\mu) = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x_\mu} = 0$$

$\Rightarrow \dfrac{d}{d\lambda}(p_\mu) = 0$ or $p_\mu = \text{const.}$ along geodesic

I-D Motion ~~[illegible]~~

~~$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$~~

~~In this case $p_x \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow p_x = m\dot{x}$ and p_x is clearly linear momentum~~

~~Also $p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}}$ But $\mathcal{L} = \mathcal{L}(x,$~~

SR: $\lambda = \tau$: $\mathcal{L} = c\sqrt{\left(\dfrac{dt}{d\tau}\right)^2 - \dfrac{1}{c^2}\left(\dfrac{dx}{d\tau}\right)^2}$

Recall $\dfrac{dx}{d\tau} = \text{const.}$

$p_x \stackrel{d}{=} m\dfrac{dx}{d\tau}$ also conserved

$p_x = m\dfrac{dx}{dt}\dfrac{dt}{d\tau} = m v_x \gamma$ also conserved

Recall: $p_x \to \infty$ as $v_x \to c$

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = c \times \frac{2\left(\frac{dt}{d\tau}\right)}{2\sqrt{}} = c\left(\frac{dt}{d\tau}\right) = \frac{E}{mc}$$

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$$p_t \equiv \frac{E}{mc} = c\left(\frac{dt}{d\tau}\right) = c\gamma \Rightarrow \boxed{\frac{dt}{d\tau} = \frac{E}{mc^2} = \gamma}$$

As a result: $1 = \sqrt{\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{d\tau}\right)^2}$

becomes $1 = \left(\frac{E}{mc^2}\right)^2 - \frac{1}{c^2}\left(\frac{p_x}{m}\right)^2$ (mult by m^2c^4)

or $(mc^2)^2 = E^2 - c^2p_x^2 \Rightarrow \boxed{E = \sqrt{(mc^2)^2 + c^2p_x^2}}$

Thus a free particle has energy even when at rest: leads to concept of rest-mass energy: Since $p_x = \gamma m v_x$

$$E = \gamma mc^2 \quad \left\{ \begin{aligned} E^2 &= (mc^2)^2 + \gamma^2 m^2 v_x^2 \\ &= (mc^2)^2\left[1 + \left(\tfrac{v}{c}\right)^2\gamma^2\right] = \gamma^2(mc^2)^2 \\ \therefore\ & \boxed{E = \gamma mc^2} \end{aligned} \right.$$

Newtonian theory:

For a free particle: Total energy $E = K = \frac{1}{2}mv^2$
Therefore, in limit $v \to 0$, $E \to 0$.

SR: By contrast, in SR $E = \gamma mc^2$,
Therefore, in limit $v \to 0$, $E = mc^2$

In other words, a free particle sitting still has energy: that energy can potentially be converted into kinetic energy or radiant energy by physical processes: fission reactions.

Kinetic Energy: In SR, $K = E - mc^2$

Therefore $K = (\gamma - 1)mc^2$

In limit $(v/c) \ll 1$, $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \simeq 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots$

thus $K \simeq \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \cdots - 1\right)mc^2 \simeq \frac{1}{2}mv^2$

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Geodesics in Schwarzschild metric

$$-ds^2 = \left(1 - \frac{2GM/c^2}{r}\right)c^2dt^2 - \left(1 - \frac{2GM/c^2}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

This metric is so called "vacuum" solution to Einstein field equations. This is because it describes spacetime outside spherically symmetric mass distribution. By contrast interior solution (described later) corresponds to spacetime in which mass is present. Although ⊙

Schwarzschild then describes sun, black-hole, ... or any spacetime outside spherical mass distribution. The only difference is ratio $\frac{2GM/c^2}{r}$. $\begin{cases} \text{sun} & \sim 10^{-6} \\ \text{BH} & \sim 1 \end{cases}$

Geodesics

Let's do time-like and lightlike geodesics at same time. For this reason: Lagrangian: (otherwise divide by 0)

$\mathcal{L} = -\left(\frac{ds}{d\lambda}\right)^2$ rather than $\mathcal{L} = \sqrt{\left(\frac{ds}{d\lambda}\right)^2}$ which works only for timelike.

Therefore:

$$\boxed{\mathcal{L} = \left(1 - \frac{2GM/c^2}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{2GM/c^2}{r}\right)} - r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)} \quad (1)$$

where $\dot{t} = dt/d\lambda$; $\dot{r} = dr/d\lambda$, $\dot{\theta} = d\theta/d\lambda$, $\dot{\phi} = d\phi/d\lambda$

Time-like: Let $\lambda = \tau$ (proper time) : $\mathcal{L} = -\left(\frac{ds}{d\tau}\right)^2 = c^2$ (1a)

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λ : $\mathcal{L} = 0$: Lightlike

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$\boxed{BH\text{-}77}$

Lagrange Equations

[t] : $$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0$$

Since $\frac{\partial \mathcal{L}}{\partial t} = 0$, $\frac{\partial \mathcal{L}}{\partial \dot{t}} = \text{const.}$ {from eq. (A) on BH-76}

$$\therefore \left(1 - \frac{2GM/c^2}{r}\right)(2c^2\dot{t}) = \text{const.}$$

or ① $$\boxed{\left(1 - \frac{2GM/c^2}{r}\right)\dot{t} = E}$$

{const. E related to energy (dimensionless)}

as $r \to \infty$ $E = \dot{t} = \gamma = \frac{\mathcal{E}}{mc^2}$ (phys. energy)

[φ] $$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Since $\frac{\partial \mathcal{L}}{\partial \phi} = 0$, $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const}$

$$\therefore -r^2 \sin^2\theta \,(2\dot{\phi}) = \text{const}$$

or ② $$\boxed{r^2 \sin^2\theta\, \dot{\phi} = L}$$ {L related to angular momentum}

Orbital plane : Turns out, because of spherical symmetry motion must occur in const. orbital plane. Without loss of generality, let's choose that plane to be $\theta = \pi/2$, equator. As a result. Eq. (2) becomes $\boxed{\dot{\theta} = 0}$

$$\boxed{r^2\dot{\phi} = L} \quad (2a)$$

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[r] this is complicated. In fact we don't need it. Rather we shall use condition:

$$\mathcal{L} = K \quad \text{where } K = \begin{cases} c^2 & : \text{timelike} \\ 0 & : \text{lightlike} \end{cases}$$

That is:

$$\boxed{\left(1 - \frac{2GM/c^2}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{2GM/c^2}{r}\right)} - r^2\dot{\phi}^2 = K} \quad (3)$$

Now: Combine eqs. (1), (2a), and (3)

Simplify; Let $A \equiv 1 - \frac{2GM/c^2}{r}$

$$\therefore \quad K = A c^2 \dot{t}^2 - \frac{\dot{r}^2}{A} - r^2 \dot{\phi}^2$$

But eq. (1) $\Rightarrow$ $\dot{t}^2 = E^2/A^2$

eq. (2a) $\Rightarrow$ $\dot{\phi}^2 = L^2/r^4$

Therefore: $K = \frac{A c^2 E^2}{A^2} - \frac{\dot{r}^2}{A} - \frac{r^2 L^2}{r^4}$

$$K = \frac{c^2 E^2}{A} - \frac{\dot{r}^2}{A} - \frac{L^2}{r^2} \; ; \quad KA = c^2E^2 - \dot{r}^2 - \frac{L^2}{r^2}A$$

Divide by 2

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$$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2}A + \frac{KA}{2} = \frac{c^2E^2}{2}$$

$$\frac{1}{2}\dot{r}^2 + \frac{A}{2}\left(K + \frac{L^2}{r^2}\right) = \frac{c^2E^2}{2}$$

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$\frac{\dot{r}^2}{2} + V(r) = \frac{c^2E^2}{2} =$ constant

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make like energy equation? (BH-79A

or $\boxed{\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM/c^2}{r}\right)\left(\kappa + \frac{L^2}{r^2}\right) = \frac{c^2E^2}{2}}$ (4)

Eq. 4 shows radial motion of geodesic in Schwarzschild spacetime is that of unit mass particle of energy $\frac{c^2E^2}{2}$ moving in effective potential:

$$\boxed{V(r) = \frac{1}{2}\kappa - \frac{\kappa\, GM/c^2}{r} + \frac{L^2}{2r^2} - \frac{GML^2/c^2}{r^3}} \quad (4a)$$

New Feature of General Relativity:

In addition to usual Newtonian terms $\left(-\frac{\kappa GM/c^2}{r}\right.$ and centrifugal barrier term $\left.\frac{L^2}{2r^2}\right)$ we have new $-\frac{GML^2/c^2}{r^3}$ term which dominates and removes centrifugal barrier term at small r: $\frac{1}{2}\dot{r}^2 + V(r) = \frac{c^2E^2}{2}$

Pure radial orbit: $L=0$

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM/c^2}{r}\right)\kappa = \frac{c^2E^2}{2}$$

Drop spaceship from rest at $r=\infty$. Timelike $\lambda=\tau$, $\kappa=c^2$

$\mathcal{L}=\kappa \begin{cases} =c^2 & \text{timelike} \\ =0 & \text{lightlike} \end{cases}$

$\therefore$ at $r=\infty$ $\quad 0 + \frac{c^2}{2} = \frac{E^2c^2}{2} \Rightarrow \quad E = 1$

$\therefore E = \frac{\mathcal{E}}{mc^2}$ where $\mathcal{E}$ is physical energy

Thus only rest mass energy at $r\to\infty$

Also condition for escape to $r\to\infty$.

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