(1) The Dieterici equation of state is
\[ p(v - b) = RT \exp\left(-\frac{a}{vRT}\right). \]

(a) Find the critical point \((p_c, v_c, T_c)\) for this equation of state

(b) Writing \(\bar{p} = p/p_c, \bar{v} = v/v_c, \) and \(\bar{T} = T/T_c,\) rewrite the equation of state in the form \(\bar{p} = p(\bar{v}, \bar{T}).\)

(c) For the brave only! Writing \(\pi = 1 + \pi, \bar{T} = 1 + t, \) and \(\bar{v} = 1 + \epsilon,\) find \(\epsilon_{\text{liq}}(t)\) and \(\epsilon_{\text{gas}}(t)\) for \(0 < (-t) \ll 1,\) working to lowest nontrivial order in \((-t).\)

Solution:

(a) We have
\[ p = \frac{RT}{v - b} e^{-a/vRT}, \]

hence
\[ \left(\frac{\partial p}{\partial v}\right)_T = p \cdot \left\{-\frac{1}{v - b} + \frac{a}{v^2 RT}\right\}. \]

Setting the LHS of the above equation to zero, we then have
\[ \frac{v^2}{v - b} = \frac{a}{RT} \Rightarrow f(u) \equiv \frac{u^2}{u - 1} = \frac{a}{bRT}, \]

where \(u = v/b\) is dimensionless. Setting \(f'(u^*) = 0\) yields \(u^* = 2,\) hence \(f(u)\) on the interval \(u \in (1, \infty)\) has a unique global minimum at \(u = 2,\) where \(f(2) = 4.\) Thus,
\[ v_c = 2b \quad , \quad T_c = \frac{a}{4bR} \quad , \quad p_c = \frac{a}{4b^2} e^{-2}. \]

(b) In terms of the dimensionless variables \(\bar{p}, \bar{v},\) and \(\bar{T},\) the equation of state takes the form
\[ \bar{p} = \frac{\bar{T}}{2\bar{v} - 1} \exp\left(2 - \frac{2}{\bar{v} \bar{T}}\right). \]

When written in terms of the dimensionless deviations \(\pi, \epsilon,\) and \(t,\) this becomes
\[ \pi = \left(\frac{1 + t}{1 + 2\epsilon}\right) \exp\left(\frac{2(\epsilon + t + \epsilon t)}{1 + \epsilon + t + \epsilon t}\right) - 1. \]

Expanding via Taylor’s theorem, one finds
\[ \pi(\epsilon, t) = 3t - 2t\epsilon + 2t^2 - \frac{2}{3}\epsilon^3 + 2\epsilon^2 t - 4\epsilon t^2 - \frac{2}{3} t^3 + \ldots. \]
Thus,
\[ \pi_{\epsilon t} \equiv \frac{\partial^2 \pi}{\partial \epsilon \partial t} = -2 \quad , \quad \pi_{\epsilon \epsilon \epsilon} \equiv \frac{\partial^3 \pi}{\partial \epsilon^3} = -4 \, , \]
and according to the results in §7.2.2 of the Lecture Notes, we have
\[ \epsilon_{L,G} = \pm \left( \frac{6 \pi_{\epsilon t}}{\pi_{\epsilon \epsilon \epsilon}} \right)^{1/2} = \pm \left( -3\epsilon \right)^{1/2} . \]

(2) Consider a ferromagnetic spin-1 triangular lattice Ising model. The Hamiltonian is
\[ \hat{H} = -J \sum_{\langle ij \rangle} S_i^z S_j^z - H \sum_i S_i^z , \]
where \( S_i^z \in \{-1, 0, +1\} \) on each site \( i \), \( H \) is a uniform magnetic field, and where the first sum is over all links of the lattice.

(a) Derive the mean field Hamiltonian \( \hat{H}_{MF} \) for this model.
(b) Derive the free energy per site \( F/N \) within the mean field approach.
(c) Derive the self consistent equation for the local moment \( m = \langle S_i^z \rangle \).
(d) Find the critical temperature \( T_c(H = 0) \).
(e) Assuming \( |H| \ll k_B|T - T_c| \ll J \), expand the dimensionless free energy \( f = F/6Nj \) in terms of \( \theta = T/T_c \), \( h = H/k_B T_c \), and \( m \). Minimizing with respect to \( m \), find an expression for the dimensionless magnetic susceptibility \( \chi = \partial m/\partial h \) close to the critical point.

Solution:

(a) Writing \( S_i^z = m + \delta S_i^z \), where \( m = \langle S_i^z \rangle \) and expanding \( \hat{H} \) to linear order in the fluctuations \( \delta S_i^z \), we find
\[ \hat{H}_{MF} = \frac{1}{2} N z J m^2 - (H + z J m) \sum_i S_i^z , \]
where \( z = 6 \) for the triangular lattice.

(b) The free energy per site is
\[ F/N = \frac{1}{2} z J m^2 - k_B T \ln \text{Tr} \, e^{(H + z J m)S^z} \]
\[ = \frac{1}{2} z J m^2 - k_B T \ln \left\{ 1 + 2 \cosh \left( \frac{H + z J m}{k_B T} \right) \right\} . \]
(c) The mean field equation is $\partial F / \partial m = 0$, which is equivalent to $m = \langle S_i^z \rangle$. We obtain

$$m = \frac{2 \sinh \left( \frac{H + zJm}{k_B T} \right)}{1 + 2 \cosh \left( \frac{H + zJm}{k_B T} \right)}.$$ 

(d) To find $T_c$, we set $H = 0$ in the mean field equation:

$$m = \frac{2 \sinh(M \beta zm)}{1 + 2 \cosh(M \beta zm)} = \frac{2}{3} \beta zm + \mathcal{O}(m^3).$$

The critical temperature is obtained by setting the slope on the RHS of the above equation to unity. Thus,

$$T_c = \frac{2zJ}{3k_B}.$$ 

So for the triangular lattice, where $z = 6$, one has $T_c = 4J/k_B$.

(e) Scaling $T$ and $H$ as indicated, the mean field equation becomes

$$m = \frac{2 \sinh \left( \frac{(m + h)/\theta}{M \beta zm} \right)}{1 + 2 \cosh \left( \frac{(m + h)/\theta}{M \beta zm} \right)} = \frac{m + h}{\theta / \theta_c} + \cdots,$$

where $\theta_c = \frac{2}{3}$, and where we assume $\theta > \theta_c$. Solving for $m(h)$, we have

$$m = \frac{h}{1 - \frac{\theta_c}{\theta}} = \frac{\theta_c h}{\theta - \theta_c} + \mathcal{O}\left((\theta - \theta_c)^2\right).$$

Thus, $\chi = \theta_c/(\theta - \theta_c)$, which reflects the usual mean field susceptibility exponent $\gamma = 1$.

(3) Consider the antiferromagnetic quantum Heisenberg model on a bipartite lattice:

$$\mathcal{H} = J \sum_{\langle ij \rangle} S_i \cdot S_j$$

where $J > 0$ and the sum is over the links of the lattice.

(a) Break up the lattice into a dimer covering (a dimer is a pair of nearest neighbor sites). Denote one sublattice as A and the other as B. You are to develop a mean field theory of interacting dimers in the presence of a self-consistent staggered field

$$\langle S_\lambda \rangle = -\langle S_B \rangle \equiv m \hat{z}.$$ 

\footnote{There are exponentially many such dimer coverings, i.e. the number grows as $e^{aN}$ where $N$ is the number of lattice sites. Think about tiling a chessboard with dominoes. The mathematical analysis of this problem was performed by H. N. V. Temperley and M. E. Fisher, \textit{Phil. Mag.} 6, 1061 (1961).}
The mean field Hamiltonian then breaks up into a sum over dimer Hamiltonians
\[ \mathcal{H}_{\text{dimer}} = JS_A \cdot S_B + (z - 1)J \langle S_B \rangle \cdot S_A + (z - 1)J \langle S_A \rangle \cdot S_B \]
\[ = JS_A \cdot S_B - H_s (S_A^z - S_B^z) \]
where the effective staggered field is \( H_s = (z - 1)Jm \), and \( z \) is the lattice coordination number. Find the eigenvalues of the dimer Hamiltonian when \( S = \frac{1}{2} \).

(b) Define \( x = 2h/J \). What is the self-consistent equation for \( x \) when \( T = 0 \)? Under what conditions is there a nontrivial solution? What then is the self-consistent staggered magnetization?

**Hint:** Write your mean field Hamiltonian as a \( 4 \times 4 \) matrix using the basis states \( |↑↑\rangle \), \( |↑↓\rangle \), \( |↓↑\rangle \), and \( |↓↓\rangle \). It should be block diagonal in such a way which allows an easy calculation of the eigenvalues.

**Solution:**

(a) The mean field Hamiltonian,
\[ \mathcal{H} = JS_A \cdot S_B - H_s (S_A^z - S_B^z) , \]
is written in matrix form (for \( S = \frac{1}{2} \)) as
\[
\mathcal{H} = \begin{pmatrix}
\frac{1}{4}J & 0 & 0 & 0 \\
0 & -\frac{1}{4}J - H_s & \frac{1}{2}J & 0 \\
0 & \frac{1}{2}J & \frac{1}{4}J + H_s & 0 \\
0 & 0 & 0 & \frac{1}{4}J \\
\end{pmatrix}.
\]
Clearly the states \( |↑↑\rangle \) and \( |↓↓\rangle \) are eigenstates of \( \mathcal{H} \) with eigenvalues \( \frac{1}{4}J \). The other two eigenvalues are easily found to be
\[ \lambda_\pm = -\frac{1}{4}J \pm \sqrt{H_s^2 + \frac{1}{4}J^2} , \]

(b) The ground state eigenvector is then
\[ |\Psi_0\rangle = \alpha |↑↓\rangle + \beta |↓↑\rangle , \]
with
\[ \frac{\beta}{\alpha} = x - \sqrt{1 + x^2} , \]
with \( x = 2H_s/J \). The staggered moment is then
\[ m = \langle S_A^z \rangle = \frac{1}{2} \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2 + |\beta|^2} = \frac{x}{2\sqrt{1 + x^2}} . \]
Since $m = \frac{1}{2} x/(z - 1)$, we have $\sqrt{1 + x^2} = (z - 1)$, or

$$m = \frac{\sqrt{z(z - 2)}}{2(z - 1)}$$

for the staggered magnetization. For $z = 4$ (square lattice) we find $m = \frac{2\sqrt{2}}{3} \approx 0.471$, which is 94% of the full moment $S = \frac{1}{2}$. Spin wave theory gives $m \approx S - 0.19 \approx 0.31$, which is only 62% of the full moment for $S = \frac{1}{2}$.

Equivalently, we can compute the total energy per dipole,

$$\mathcal{E} = (z - 1)Jm^2 + \lambda_\_$$

where the first term is the mean field energy of the $(z - 1)$ links per site treated in the mean field approximation. Minimizing with respect to $m$,

$$\frac{\partial \mathcal{E}}{\partial m} = 2(z - 1)Jm - \frac{(z - 1)^2 J^2 m}{\sqrt{(z - 1)^2 J^2 m^2 + \frac{1}{4} J^2}}.$$  \hspace{1cm} (1)

Solving for $m$, we recover the result $m = \sqrt{z(z - 2)}/2(z - 1)$. 

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