Physics 4E **Quiz 3 Solutions** Spring 2012

Problem 1

$$E_n = E_1 n^2 \quad , \text{with } E_1 = \frac{\hbar^2 \pi^2}{2 m_e L^2}$$

$$hf = E_n - E_{n-1} = E_1 \left(n^2 - (n-1)^2 \right) = E_1 (2n-1) = \frac{\hbar^2 \pi^2}{2 m_e L^2} (2n-1)$$

$$\Rightarrow f = \frac{\hbar^2 \pi^2}{2 m_e L^2 h} (2n-1) = \frac{\cancel{\hbar} \cdot \hbar \cdot \pi^{\cancel{2}}}{2\cancel{\pi} \cdot 2 m_e L^2 \cancel{\hbar}} (2n-1) \Rightarrow \boxed{f = \frac{\pi \hbar}{4 m_e L^2} (2n-1)} \ (a)$$

(b) $E = \frac{p^2}{2m_e} \Rightarrow p^2 = 2 m_e E \Rightarrow$ in n-th state, $p^2 = 2 m_e E_n \Rightarrow$

$$p^2 = 2 m_e E_1 n^2 = \frac{\hbar^2 \pi^2}{L^2} n^2 \Rightarrow \quad p = \frac{\pi \hbar}{L} n$$

Now $p = m_e v \Rightarrow \quad v = \frac{\pi \hbar}{m_e L} n$; electron is moving back and forth with speed $v \Rightarrow$ travels distance $2L$ in time $t = \frac{2L}{v} = \frac{2L^2 m_e}{\pi \hbar n}$

$\Rightarrow$ the classical frequency of oscillation is

$$\boxed{f = \frac{1}{t} = \frac{\pi \hbar}{2 m_e L^2} n}$$

(c) For large n, $2n-1 \approx 2n$, so the quantum frequency is

$$f = \frac{\pi \hbar}{4 m_e L^2} (2n-1) \approx \frac{\pi \hbar}{2 m_e L^2} n$$

$\Rightarrow$ agrees with classical frequency for large n, as expected from the correspondence principle.

Problem 2

$E_n = \hbar\omega(n+\frac{1}{2})$; first excited state is $n=1 \Rightarrow n+\frac{1}{2} = \frac{3}{2} \Rightarrow$

$\Rightarrow E_1 = \frac{3}{2}\hbar\omega = \frac{1}{2} m_e \omega^2 A_1^2$, where A_1 is the classical amplitude

$\Rightarrow \hbar^2 E_1 = \frac{1}{2} m_e (\hbar\omega)^2 A_1^2 = \frac{1}{2} m_e \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{2}\hbar\omega\right)^2 \cdot A_1^2 \Rightarrow$

$\Rightarrow \hbar^2 \cancel{E_1} = \frac{2}{9} m_e E_1^{\cancel{2}} A_1^2 \Rightarrow E_1 = \frac{9\hbar^2}{2 m_e A_1^2} \Rightarrow$

$\Rightarrow E_1 = 7.62\,\text{eV}\,\text{Å}^2 \cdot \frac{9}{2} \cdot \frac{1}{25\,\text{Å}^2} \Rightarrow \boxed{E_1 = 1.37\,\text{eV}}$

(b) Probability is $P(x) = |\Psi(x)|^2 = C^2 x^2 e^{-\frac{m_e\omega}{\hbar} x^2}$

Find max: $P'(x_m) = 0 = 2x_m - x_m^2 \cdot \frac{m_e\omega}{\hbar} \cdot 2x_m \Rightarrow \boxed{x_m = \left(\frac{\hbar}{m_e\omega}\right)^{1/2}}$

(c) The classical amplitude is:

$\frac{3}{2}\hbar\omega = \frac{1}{2} m_e \omega^2 A_1^2 \Rightarrow A_1^2 = \frac{\cancel{2}}{m_e\omega^{\cancel{2}}} \frac{3}{\cancel{2}} \hbar\cancel{\omega} = \frac{3\hbar}{m_e\omega}$

$\Rightarrow A = \sqrt{3}\left(\frac{\hbar}{m_e\omega}\right)^{1/2} \Rightarrow X_m = \frac{A}{\sqrt{3}}$, $A = \sqrt{3}\, X_m$

Ratio of probabilities: since $P(x) \propto x^2 e^{-x^2/x_m^2} \Rightarrow$

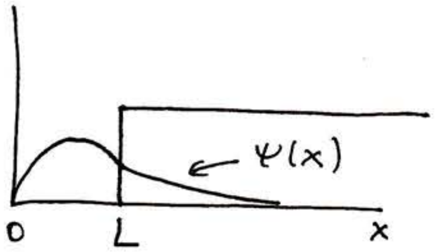
$$\frac{P(X_m)}{P(A_1)} = \frac{e^{-X_m^2/X_m^2}}{e^{-3X_m^2/X_m^2}} \frac{X_m^2}{3X_m^2} = \frac{e^3}{3} = 6.7$$

$\Rightarrow$ electron is 6.7 times more likely to be at X_m than at the classical amplitude

Problem 3

(i) $\psi(x) = A\sin kx + B\cos kx$

(ii) $\psi(x) = Ce^{-\alpha x} + De^{\alpha x}$



(a) Since $V(x) = \infty$ for $x<0$ $\Rightarrow$ $\psi(x) = 0$ for $x<0$ $\Rightarrow$ by continuity, $\psi(x=0) = 0$ $\Rightarrow$ $\boxed{B=0}$

Since wavefunction has to be normalizable and $e^{\alpha x} \xrightarrow[x\to\infty]{} \infty$ $\Rightarrow$ $\boxed{D=0}$

(b) Since wavefunction satisfies Schrödinger eq. in all regions

$$k = \left(\frac{2m_e}{\hbar^2}E\right)^{1/2} = \left(\frac{2}{7.62}\cdot 3\right)^{1/2} \mathring{A}^{-1} = \boxed{0.887\,\mathring{A}^{-1}}\ ; \quad \boxed{\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} = 1.02\,\mathring{A}^{-1}}$$

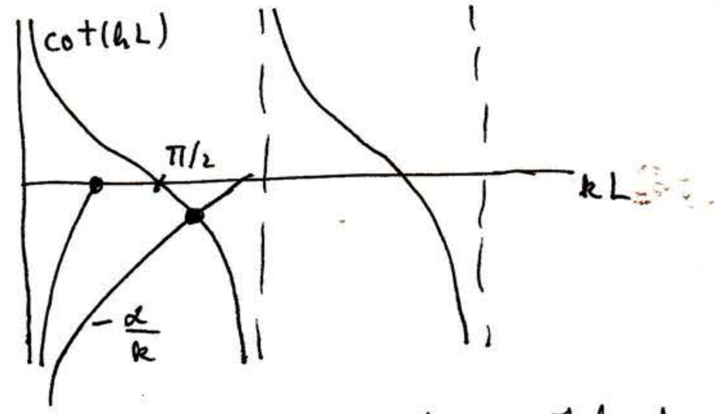
(c) See graph of $\psi(x)$ in figure above.

(d) $\psi(x) = A\sin kx$ for $0<x<L$, $\psi(x) = Ce^{-\alpha x}$ for $x>L$ $\Rightarrow$

Continuity $\Rightarrow$ $A\sin kL = Ce^{-\alpha L}$

cont. of ψ' $\Rightarrow$ $kA\cos kL = -\alpha Ce^{-\alpha L}$

$\Rightarrow$ $\boxed{\cot(kL) = -\frac{\alpha}{k}}$



The figure shows two examples of possible functions $-\alpha/k$. One of them intersects $\cot(kL)$ $\Rightarrow$ there is a solution the other one does not $\Rightarrow$ no solution.

$\therefore$ there is no solution if V_0 is very small