6-1 (a) Not acceptable – diverges as $x \to \infty$.

(b) Acceptable.

(c) Acceptable.

(d) Not acceptable – not a single-valued function.

(e) Not acceptable – the wave is discontinuous (as is the slope).

6-2 (a) Normalization requires

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{A^2}{2}\right) \int_{-\frac{L}{4}}^{\frac{L}{4}} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

so $A = \frac{2}{\sqrt{L}}$.

(b) $$P = \int_0^{\frac{L}{8}} |\psi|^2 dx = A^2 \int_0^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{4}{L}\right)\left(\frac{1}{2}\right) \int_0^{\frac{L}{8}} \left(1 + \cos\left(\frac{4\pi x}{L}\right) dx\right)$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{8}\right) + \left(\frac{2}{L}\right)\left(\frac{L}{4\pi}\right) \sin\left(\frac{4\pi x}{L}\right)\Big|_0^{\frac{L}{8}} = \frac{1}{4} + \frac{1}{2\pi} = 0.409$$

6-3 (a) $A \sin\left(\frac{2\pi x}{\lambda}\right) = A \sin\left(5 \times 10^{10} x\right)$ so $\left(\frac{2\pi}{\lambda}\right) = 5 \times 10^{10}\ \text{m}^{-1}$, $\lambda = \frac{2\pi}{5 \times 10^{10}} = 1.26 \times 10^{-10}\ \text{m}$.

(b) $$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\ \text{Js}}{1.26 \times 10^{-10}\ \text{m}} = 5.26 \times 10^{-24}\ \text{kg m/s}$$

6-6

$$\psi(x) = A \cos kx + B \sin kx$$

$$\frac{\partial \psi}{\partial x} = -kA \sin kx + kB \cos kx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 A \cos kx - k^2 B \sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E - U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A \cos kx + B \sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E - U)\psi$ or

$$-k^2 (A \cos kx + B \sin kx) = \left(\frac{-2mE}{\hbar^2}\right)(A \cos kx + B \sin kx).$$

Therefore $E = \frac{\hbar^2 k^2}{2m}$.

6-9 $E_n = \frac{n^2h^2}{8mL^2}$, so $\Delta E = E_2 - E_1 = \frac{3h^2}{8mL^2}$

$$\Delta E = (3)\frac{(1240 \text{ eV nm}/c)^2}{8\left(938.28\times 10^6 \text{ eV}/c^2\right)\left(10^{-5} \text{ nm}\right)^2} = 6.14 \text{ MeV}$$

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV nm}}{6.14\times 10^6 \text{ eV}} = 2.02\times 10^{-4} \text{ nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-11 In the present case, the box is displaced from (0, L) by $\frac{L}{2}$. Accordingly, we may obtain the wavefunctions by replacing x with $x-\frac{L}{2}$ in the wavefunctions of Equation 6.18. Using

$$\sin\left[\left(\frac{n\pi}{L}\right)\left(x-\frac{L}{2}\right)\right] = \sin\left[\left(\frac{n\pi x}{L}\right)-\frac{n\pi}{2}\right] = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi}{2}\right)-\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi}{2}\right)$$

we get for $-\frac{L}{2}\le x\le\frac{L}{2}$

$$\psi_1(x) = \left(\frac{2}{L}\right)^{1/2}\cos\left(\frac{\pi x}{L}\right);\ P_1(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{\pi x}{L}\right)$$
$$\psi_2(x) = \left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{2\pi x}{L}\right);\ P_2(x) = \left(\frac{2}{L}\right)\sin^2\left(\frac{2\pi x}{L}\right)$$
$$\psi_3(x) = \left(\frac{2}{L}\right)^{1/2}\cos\left(\frac{3\pi x}{L}\right);\ P_3(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{3\pi x}{L}\right)$$

6-12 $\Delta E = \frac{hc}{\lambda} = \left(\frac{h^2}{8mL^2}\right)\left[2^2-1^2\right]$ and $L = \left[\frac{(3/8)h\lambda}{mc}\right]^{1/2} = 7.93\times 10^{-10} \text{ m} = 7.93 \text{ Å}$.

6-16 (a) $\psi(x) = A\sin\left(\frac{\pi x}{L}\right)$, $L = 3$ Å. Normalization requires

$$1 = \int_0^L |\psi|^2 \, dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{LA^2}{2}$$

so $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \int_0^{L/3} |\psi|^2 \, dx = \left(\frac{2}{L}\right) \int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{\pi} \int_0^{\pi/3} \sin^2 \phi d\phi = \frac{2}{\pi}\left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8}\right] = 0.195\,5 .$$

(b) $\psi = A\sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \frac{2}{L} \int_0^{L/3} \sin^2\left(\frac{100\pi x}{L}\right) dx = \frac{2}{L}\left(\frac{L}{100\pi}\right) \int_0^{100\pi/3} \sin^2 \phi d\phi = \frac{1}{50\pi}\left[\frac{100\pi}{6} - \frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.331\,9$$

6-23 Inside the well, the particle is free and the Schrödinger waveform is trigonometric with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = A\sin kx + B\cos kx \qquad 0 \le x \le L .$$

The infinite wall at $x=0$ requires $\psi(0)=B=0$. Beyond $x=L$, $U(x)=U$ and the Schrödinger equation $\frac{d^2\psi}{dx^2}=\left(\frac{2m}{\hbar^2}\right)\{U-E\}\psi(x)$, which has exponential solutions for $E<U$

$$\psi(x)=Ce^{-\alpha x}+De^{+\alpha x}, \qquad x>L$$

where $\alpha=\left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. To keep ψ bounded at $x=\infty$ we must take $D=0$. At $x=L$, continuity of ψ and $\frac{d\psi}{dx}$ demands

$$A\sin kL=Ce^{-\alpha L}$$
$$kA\cos kL=-\alpha Ce^{-\alpha L}$$

Dividing one by the other gives an equation for the allowed particle energies: $k\cot kL=-\alpha$. The dependence on *E* (or *k*) is made more explicit by noting that $k^2+\alpha^2=\frac{2mU}{\hbar^2}$, which allows the energy condition to be written $k\cot kL=-\left[\left(\frac{2mU}{\hbar^2}\right)-k^2\right]^{1/2}$. Multiplying by *L*, squaring the result, and using $\cot^2\theta+1=\csc^2\theta$ gives $(kL)^2\csc^2(kL)=\frac{2mUL^2}{\hbar^2}$ from which we obtain $\frac{kL}{\sin kL}=\left(\frac{2mUL^2}{\hbar^2}\right)^{1/2}$. Since $\frac{\theta}{\sin\theta}$ is never smaller than unity for *any* value of θ, there can be no bound state energies if $\frac{2mUL^2}{\hbar^2}<1$.

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2}=\left(\frac{2m}{\hbar^2}\right)\{U(x)-E\}\psi(x)$ with $U(x)=\frac{1}{2}m\omega^2x^2$ for the quantum oscillator. Differentiating $\psi(x)=Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx}=-2\alpha x\psi(x)+C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2}=-\frac{2\alpha x d\psi}{dx}-2\alpha\psi(x)-(2\alpha x)Ce^{-\alpha x^2}=(2\alpha x)^2\psi(x)-6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$. Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and $E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2C^2 \int x^2 e^{-2\alpha x^2} dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$. For the calculation of the average position $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi_0(x)|^2 dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand $x^2|\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\langle x^2\rangle = 2C_0^2 \int_0^{\infty} x^2 e^{-ax^2} dx = 2C_0^2\left(\frac{1}{4a}\right)\left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\langle x^2\rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.