Establishing Relationships, Confidence of Data, Propagation of Uncertainties for Racket Balls and Rods

Lecture # 4

Physics 2BL

Spring 2012

Outline

- Testing models, modifying models
- Review of Gaussian distributions
- Rejection of data?
- Determining the relationship between measured values
- Uncertainties for lab 2
  - Propagate errors
  - Minimize errors

Models

- Model is a construction that represents a subject or imitates a system (invented, correspond closely to real world)
- Used to predict other behaviors (extrapolation)
- Provides context for measurements and design of experiments (must be testable)
  - guide to features of significance during observation

Testing model

- Models must be consistent with data
- Decide between competing models
  - elaboration: extend model to region of disagreement
  - precision: prefer model that is more precise
  - simplicity: Ockham's razor

How Models Change

- If models disagree with observation, we change the model
  - Refine - add to existing structure
  - Restrict - limit scope of utility
  - Replace - start over

Refining

- Original model consistent with observations, but not complete
- Extend model to account for new observations
- May include new concepts

  e.g. Model of interaction between charged objects; to include interactions between charged & uncharged add concept of induced charge

Restriction

- New model correct in situations where old isn't
- New model agrees w/ old over some range

⇨ Old still useful in limited range

e.g. General relativity vs. classical gravitational theory

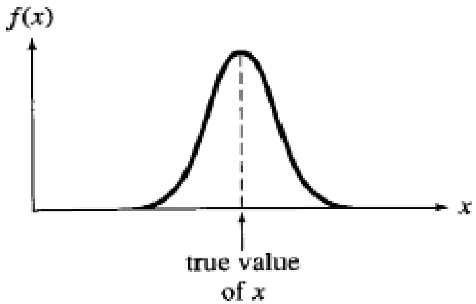
Replacement

- Old model can’t be extended consistently
- Replace entire model

⇨ Earlier observations provide limits for new model

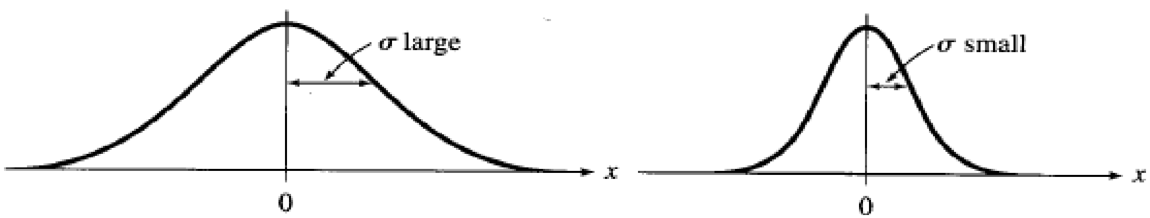
e.g. Geocentric vs heliocentric models for solar system

The Gauss, or Normal Distribution



the limiting distribution for a measurement subject to many small random errors is bell shaped and centered on the true value of x

the mathematical function that describes the bell-shape curve is called the <u>normal distribution</u>, or <u>Gauss function</u>



prototype function

$$e^{-x^2/2\sigma^2}$$

$$e^{-(x-X)^2/2\sigma^2}$$

σ – width parameter

X – true value of x

The Gaussian Distribution

- A bell-shaped distribution curve that approximates many physical phenomena - even when the underlying physics is not known.
- Assumes that many small, independent effects are additively contributing to each observation.
- Defined by two parameters: Location and scale, i.e., mean and standard deviation (or variance, σ^2).
- Importance due (in part) to central-limit theorem:

  The sum of a large number of independent and identically-distributed random variables will be approximately normally distributed (i.e.,following a Gaussian distribution, or bell-shaped curve) if the random variables have a finite variance.

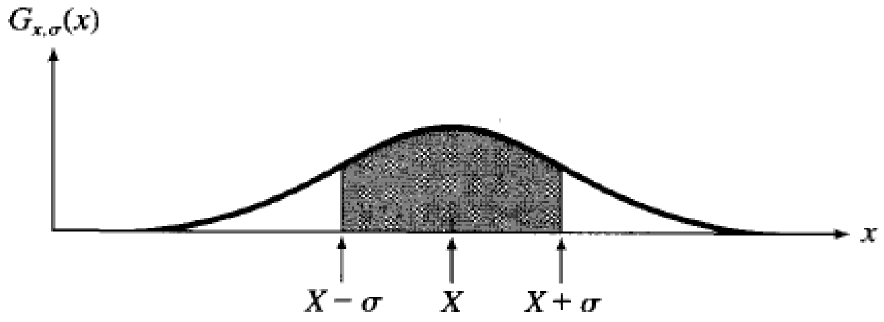
The Gauss, or Normal Distribution

normalize $e^{-(x-X)^2/2\sigma^2}$ $\longrightarrow$ $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

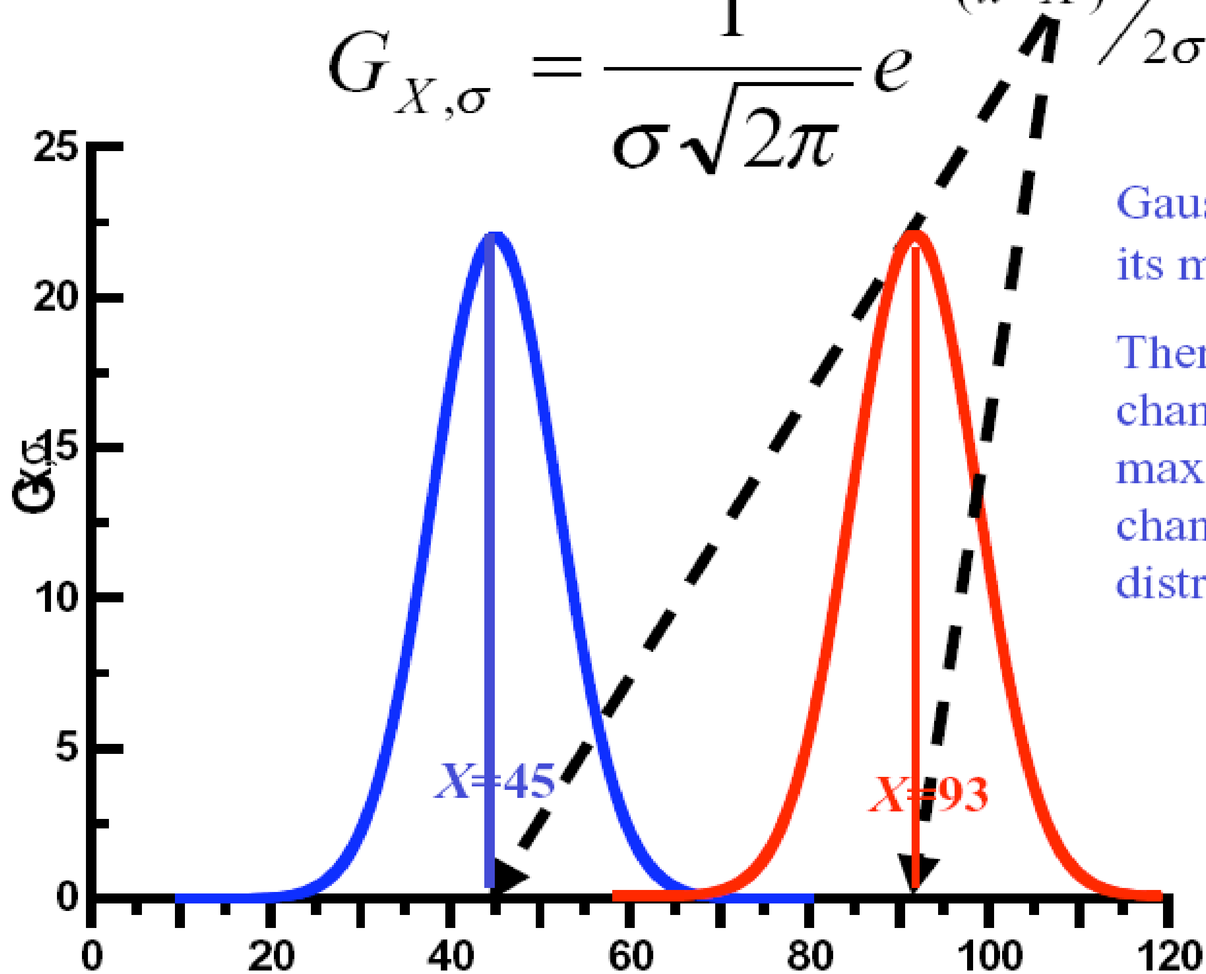
standard deviation σ_x = width parameter of the Gauss function σ

the mean value of x = true value X



Gauss distribution: changing X

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

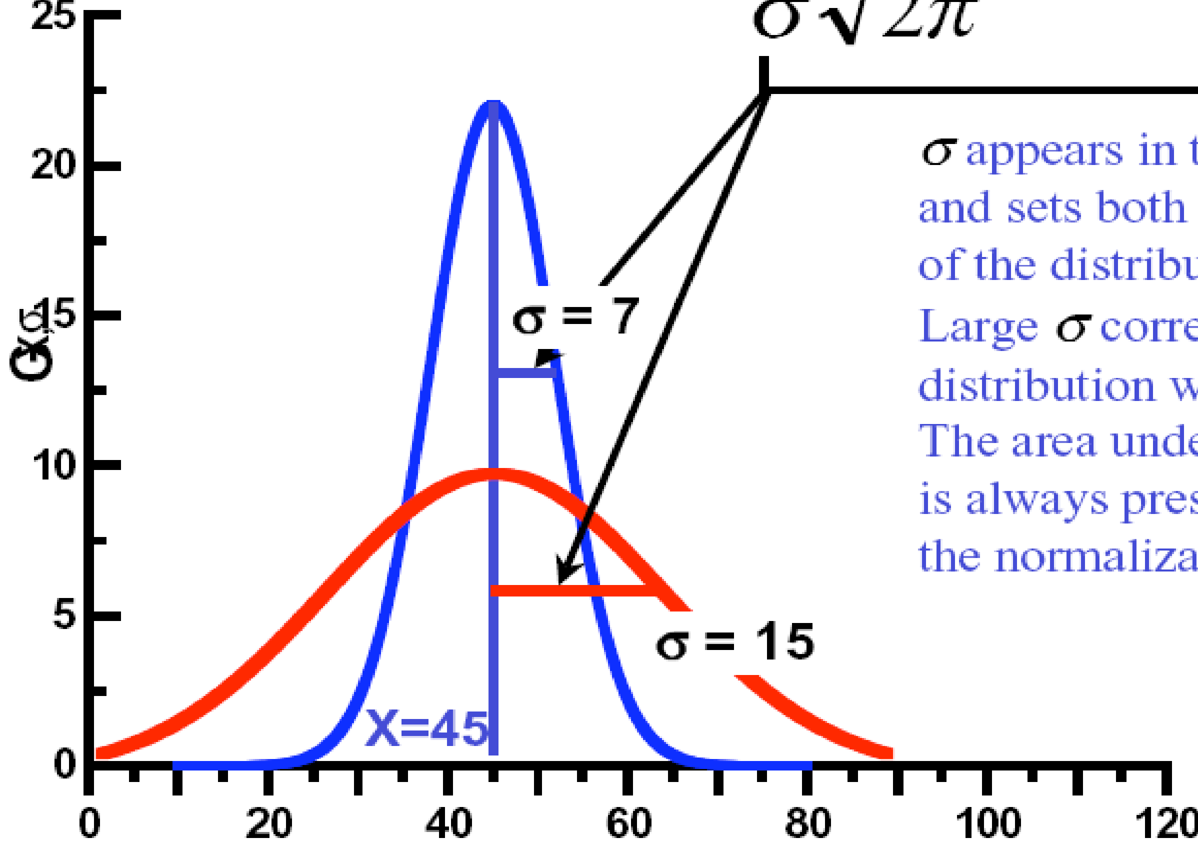


Gauss distribution has its maximum at $x = X$.

Therefore, changing X, changes position of the maximum, but does not change the shape of the distribution.

Gauss distribution: changing σ

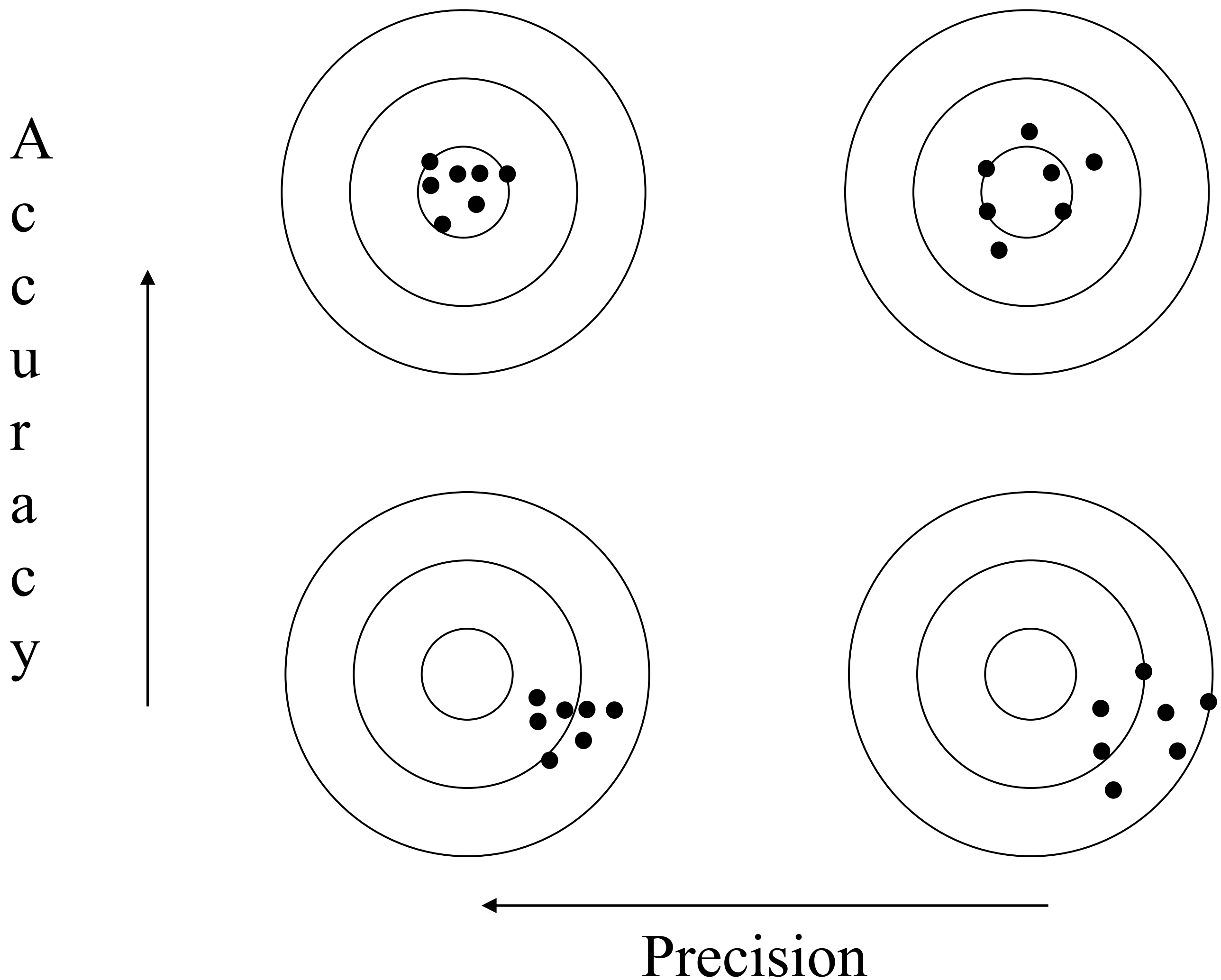
$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$



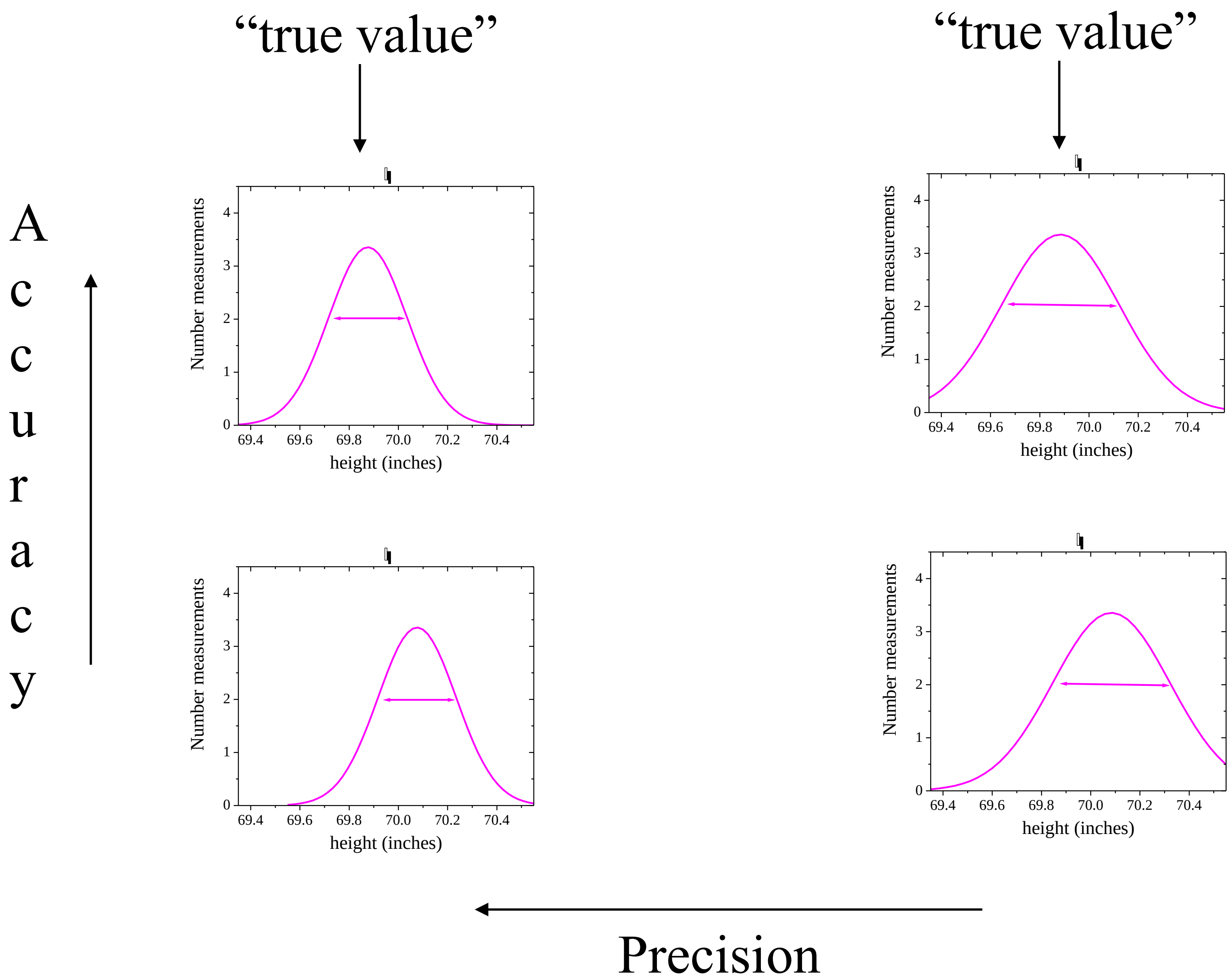
σ appears in the equation twice and sets both height and width of the distribution.
Large σ corresponds to a wider distribution with a lower peak.
The area under the distribution is always preserved, because of the normalization

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Accuracy vs. Precision



Accuracy vs. Precision

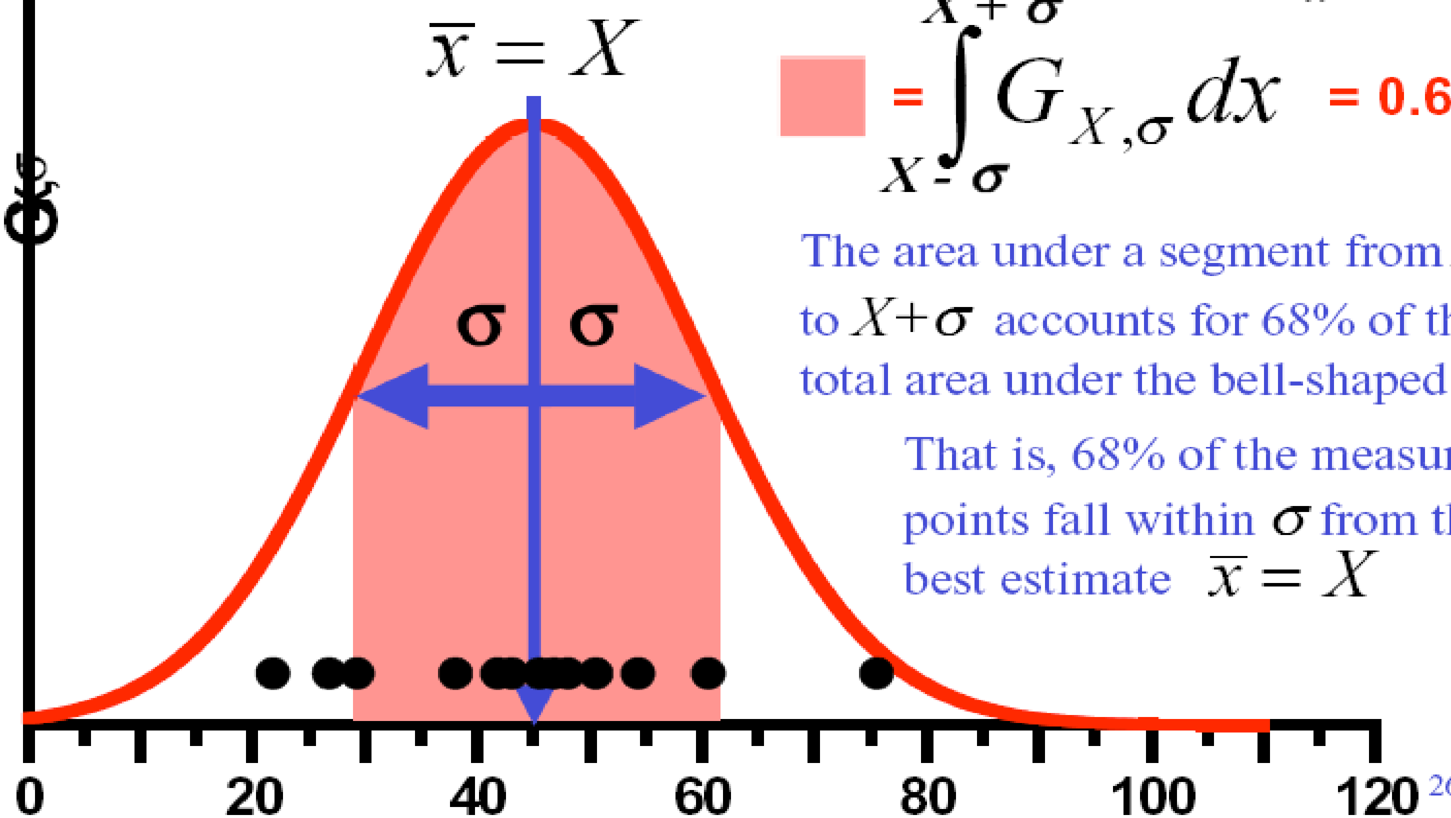


Gauss distribution: the meaning of σ

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$

$$\sigma_x = \sigma$$

$$= \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}\, dx = 0.68$$



The area under a segment from $X-\sigma$ to $X+\sigma$ accounts for 68% of the total area under the bell-shaped curve.

That is, 68% of the measured points fall within σ from the best estimate $\bar{x} = X$

Yagil

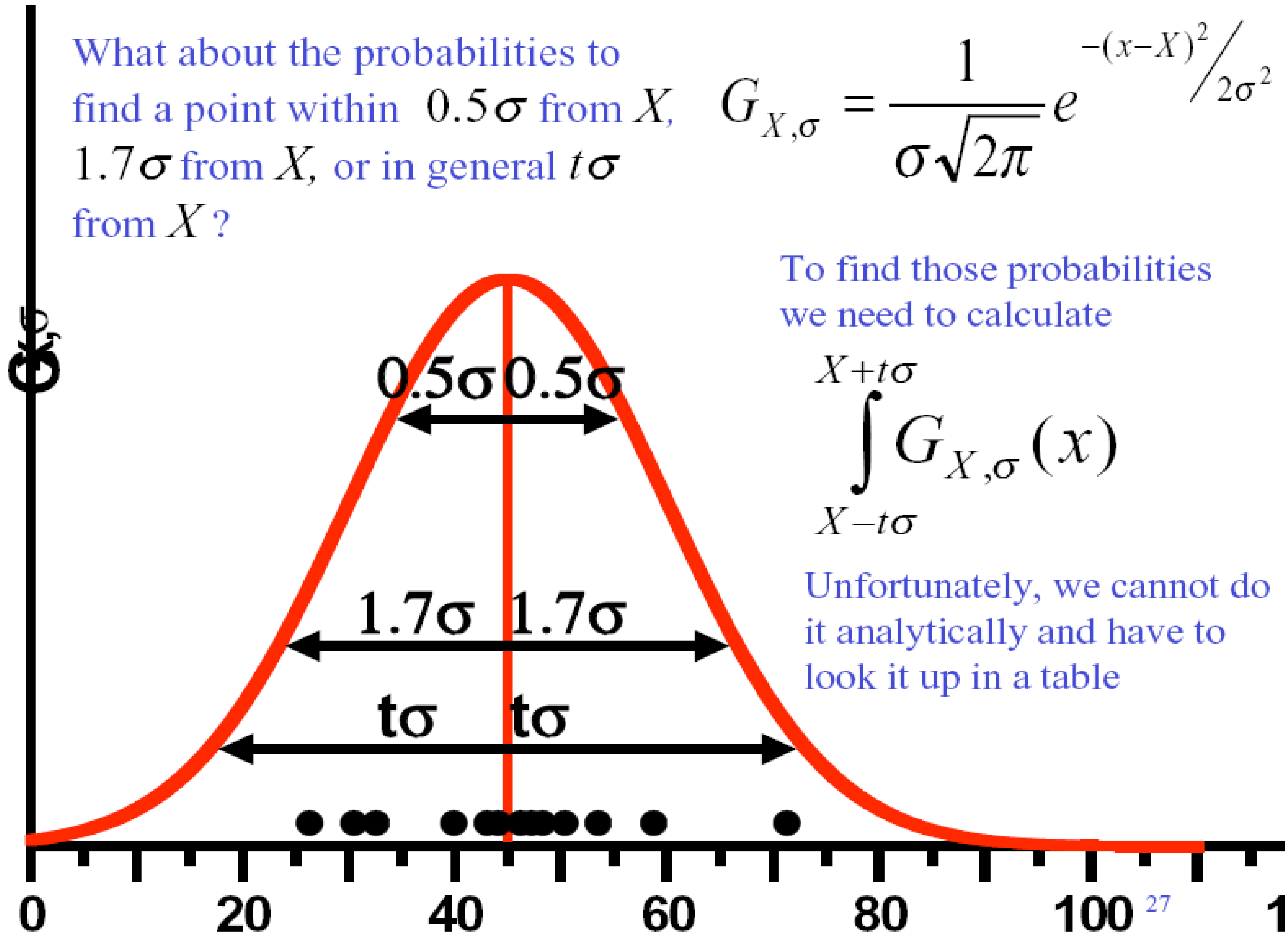
What about the probabilities to find a point within 0.5σ from X, 1.7σ from X, or in general $t\sigma$ from X?

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$

To find those probabilities we need to calculate

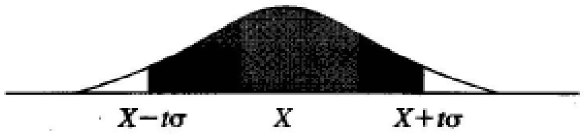
$$\int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)$$

Unfortunately, we cannot do it analytically and have to look it up in a table



Yagil

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



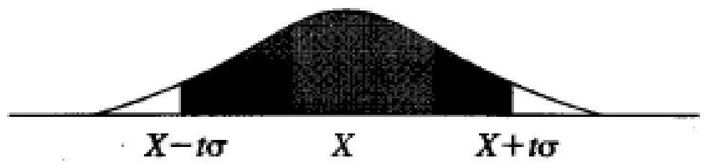
t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80

$t = 1$

p. 287 Taylor

Clicker Question 4

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



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0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78

Referring to the table above, what is the probability that a data point differs by 0.59 σ or greater?

(A) 38
(B) 44
(C) 56
(D) 62

Compatibility of a measured result(s): t-score

- Best estimate of x: $x_{best} \pm \sigma_{\bar{X}}$

- Compare with expected answer x_{exp} and compute t-score:

$$t \equiv \frac{\left| x_{best} - x_{\exp ected} \right|}{\sigma_X}$$

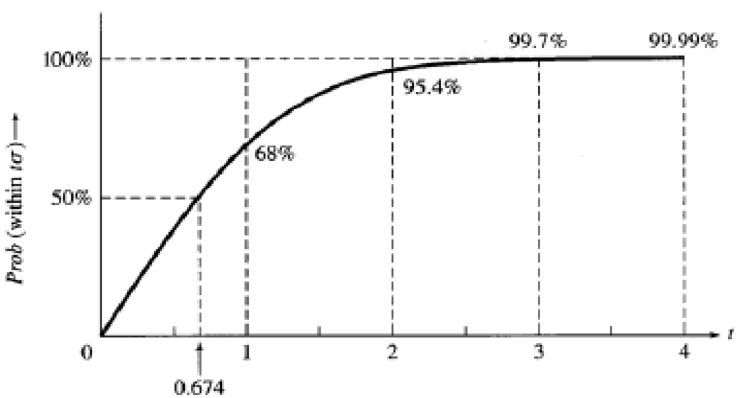
- This is the number of standard deviations that x_{best} differs from x_{exp}.
- Therefore,the probability of obtaining an answer that differs from x_{exp} by t or more standard deviations is:

Prob(outside tσ) = 1-Prob(within tσ))

"Acceptability" of a measured result
Conventions

• Large probability means likely outcome and hence reasonable discrepancy.
• "reasonable" is a matter of convention…

• We define:



erf(t) – error function

$< 5\ \%$ - significant discrepancy, $t > 1.96$

$< 1\ \%$ - highly significant discrepancy, $t > 2.58$

boundary for unreasonable improbability

If the discrepancy is beyond the **chosen** boundary for unreasonable improbability, ==> the theory and the measurement are incompatible (at the stated level)

Example: Confidence Level

Two students measure the radius of a planet.

• Student A gets R=9000 km and estimates an error of σ= 600 km

• Student B gets R=6000 km with an error of σ=1000 km

• What is the probability that the two measurements would disagree by more than this (given the error estimates)?

==> Define the quantity $q = R_A - R_B = 3000$ km. The expected q is zero.

Use propagation of errors to determine the error on q.

$$\sigma_q = \sqrt{\sigma_A^2 + \sigma_B^2} = 1170 \text{ km}$$

• Compute t the number of standard deviations from the expected q.

$$t = \frac{q}{\sigma_q} = \frac{9000 - 6000}{1170} = 2.56$$

• Now we look at Table A ==> 2.56 σ corresponds to 98.95%

So, The probability to get a worse result is 1.05% (=100-98.95)

We call this the Confidence Level, and this is a bad one.

Rejection of Data ?

Chapter 6

- Consider series – 3.8s, 3.5s, 3.9s, 3.9s, 3.4s, 1.8s
- Reject 1.8s ?
  - Bad measurement
  - New effect
    - Something new
- Make more measurements so that it does not matter

How different is the data point?

- From series obtain
  - $\langle x \rangle = 3.4s$
  - $\sigma = 0.8s$
- How does 1.8s data point apply?
- How far from average is it?
  - $x - \langle x \rangle = \Delta x = 1.6\ s = 2\ \sigma$
- How probable is it?
  - Prob $(|\Delta x| > 2\ \sigma) = 1 - 0.95 = 0.05$

Chauvenet's Criterion

- Given our series, what is prob of measuring a value 2 σ off ?
  - Multiply Prob by number of measurement
  - Total Prob = 6 x 0.05 = 0.3
- If chances < 50% discard

Strategy

- $t_{sus} = \Delta x$ (in σ)
- Prob of x outside Δx
- Total Prob = N x Prob
- If total Prob < 50% then reject

Refinement

- When is it useful
  - Best to identify suspect point
  - remeasure
- When not to reject data
  - When repeatable
  - May indicate insufficient model
  - Experiment may be sensitive to other effects
  - May lead to something new (an advance)

Rejection of other data points

- If more than one data point suspect, consider that model is incorrect
- Look at distribution
- Additional analysis
  - Such as χ^2 testing (chapter 12)
  - Remeasure/ repeatable
  - Determine circumstances were effect is observed.

Clicker Question 5

Suppose you roll the ball down the ramp 5 times and measure the rolling times to be [3.092 s, 3.101 s, 3.098 s, 3.095 s, 4.056 s]. For this set, the average is 3.288 s and the standard deviation is 0.4291 s. According to Chauvenet's criterion, would you be justified in rejecting the time measurement t = 4.056 s?

(A) Yes
(B) No
(C) Give your partner a time-out

(A) t-score = (4.056 s – 3.288 s) / 0.4291 s = 1.78 σ
(B) Prob within t-score = 92.5
(C) Prob outside t-score = 7.5
(D) Total prob = 5*7.5 = 37.5 %
(E) < 50%, reject

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1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
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1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34

Useful concept for complicated formula

- Often the quickest method is to calculate with the extreme values
  - $q = q(x)$
  - $q_{max} = q(\overline{x} + \delta x)$
  - $q_{min} = q(\overline{x} - \delta x)$

  $\Rightarrow \delta q = (q_{max} - q_{min})/2$ (3.39)

The Four Experiments

- **Determine the average density of the earth**

Weigh the Earth, Measure its volume

– Measure simple things like lengths and times

– Learn to estimate and propagate errors

- **Non-Destructive measurements of densities, inner structure of objects**

– Absolute measurements *vs.* Measurements of variability

– Measure moments of inertia

– Use repeated measurements to reduce random errors

- **Construct and tune a shock absorber**

– Adjust performance of a mechanical system

– Demonstrate critical damping of your shock absorber

- **Measure coulomb force and calibrate a voltmeter.**

– Reduce systematic errors in a precise measurement.

Rotational Kinematics

Linear Kinematics

$$v_f = v_i + a\Delta t$$

$$\Delta s = v_i \Delta t + \tfrac{1}{2} a (\Delta t)^2$$

$$v_f^2 = v_i^2 + 2a\Delta s$$

Rotational Kinematics

$$\omega_f = \omega_i + \alpha\Delta t$$

$$\Delta\theta = \omega_i \Delta t + \tfrac{1}{2} \alpha (\Delta t)^2$$

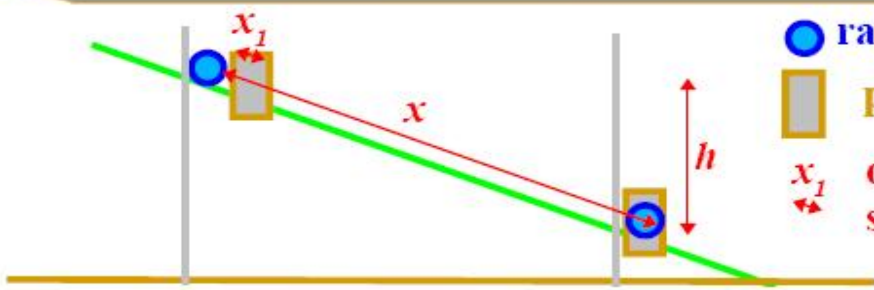
$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$s = \theta r \qquad v = \omega r \qquad a_t = \alpha r$$

Racquet Balls

We should check if the variation in *d* is much less than 10%.

Measuring I by Rolling Objects



- **racketball**
- **photogate timer**
- x_1 **distance before starting timer**

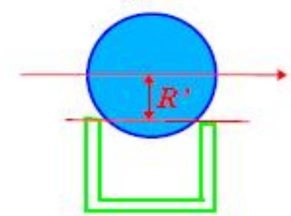
1. Measure M and R
2. Using photo gate timer measure the time, t, to travel distance x

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$ energy conservation

$$v = R'\omega$$ rolling radius

$$v = \frac{2x}{t}$$ for uniform acceleration

rolling radius R'



$$Mgh = \frac{1}{2}v^2\left(M + \frac{I}{R'^2}\right)$$

$$gh = \frac{2x^2}{t^2}\left(1 + \frac{I}{MR'^2}\right)$$

$$\frac{I}{MR'^2} = \left(\frac{ght^2}{2x^2} - 1\right)$$

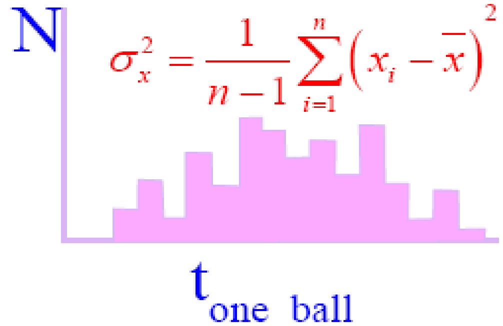
$$\tilde{I} \equiv \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right)$$

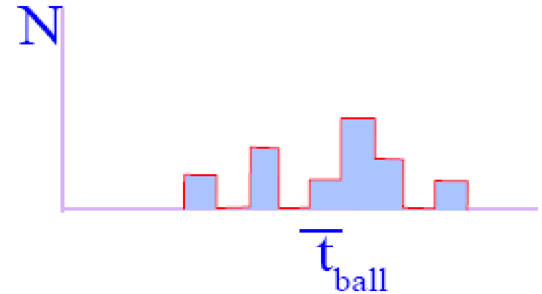
Measuring the Variation in Thickness of the Shell

- 1. Measure rolling time of one ball many times to determine the measurement error in *t*, $\sigma_{measurement}$
- 2. Measure rolling time of many balls to determine the total spread in *t*, σ_{total}
- 3. Calculate the spread in time due to ball manufacture, $\sigma_{manufacture}$, by subtracting the measurement error
- 4. Propagate error on *t* into error on *I* and then into error on thickness *d*

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

statistical analysis





$$\sigma_{total} = \sigma_{manufacture} \oplus \sigma_{measruement}$$

$\sigma_t \longrightarrow \sigma_I \longrightarrow \sigma_d$

variation in *t* → variation in *I* → variation in *d*

Propagate Error from *I* to *d*

$$I = \frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}$$

$$z \equiv \frac{r}{R} \approx \frac{28.25 - 4.5 \text{ mm}}{28.25 \text{ mm}} \approx 0.841$$

measured thickness and radius for one ball

d=4.5 mm R=28.25 mm

$d=R-r$

$$\tilde{I}(0.841) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571892$$

$$\tilde{I}(0.840) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571366$$

$$\frac{\partial z}{\partial \tilde{I}} = \frac{0.841 - 0.840}{0.571892 - 0.571366} = \frac{0.001}{0.00526} = 1.901$$

$\delta z \leftrightarrow \delta I$ numerically

$$\frac{\sigma_d}{d} = \frac{\sigma_r}{d} = \frac{R\sigma_z}{d} = \frac{R\tilde{I}}{d}\frac{\partial z}{\partial \tilde{I}}\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{(28.25 \text{ mm})(0.572)}{4.5 \text{ mm}}(1.901)\frac{\sigma_{\tilde{I}}}{\tilde{I}} = 6.826\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

Propagate Error from t to I

$$\tilde{I} = \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right) \approx 0.572$$ from previous page

$$\frac{\partial \tilde{I}}{\partial t} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)$$ compute derivative

$$\sigma_{\tilde{I}} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)\sigma_t$$ propagate error

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} = \frac{\left(\frac{ght}{x^2}\right)}{\left(\frac{ght^2}{2x^2} - 1\right)}\sigma_t \approx \frac{\left(\frac{ght}{x^2}\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t$$

work out fractional error numerically

$$\left(\frac{ght}{x^2}\right) = \frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)$$

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{\frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t = \frac{2\left(0.572 + \frac{R'^2}{R^2}\right)}{(0.572)}\frac{\sigma_t}{t} \approx 4\frac{\sigma_t}{t}$$

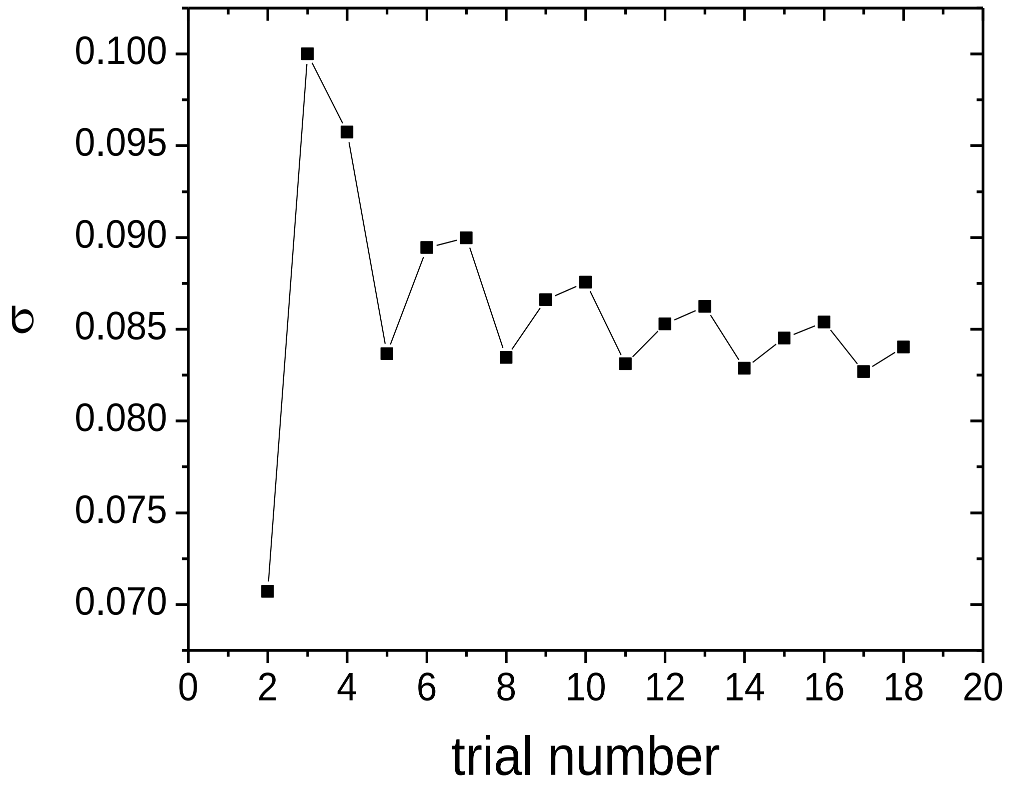
$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 4\frac{\sigma_t}{t}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 27\frac{\sigma_t}{t}$$

to get a 10% error on the thickness we need 0.37% error on the rolling time

accuracy can be improved by rolling each ball many times

Standard Deviation versus Trial Number



=STDEV(A$1:A2)

Remember

- Read lab description for experiment #2, prepare, Quiz 2
- Read Taylor Chapter 6 through 9
- Problems 6.4, 7.2, 8.6, 8.10