

8 Boltzmann Equation : Worked Examples

(8.1) Consider a monatomic ideal gas in the presence of a temperature gradient ∇T . Answer the following questions within the framework of the relaxation time approximation to the Boltzmann equation.

(a) Compute the particle current $\mathbf{j}$ and show that it vanishes.

(b) Compute the 'energy squared' current,

$$\mathbf{j}_{\varepsilon^2} = \int d^3p \, \varepsilon^2 \mathbf{v} f(\mathbf{r}, \mathbf{p}, t) \quad .$$

(c) Suppose the gas is diatomic, so $c_p = \frac{7}{2}k_B$. Show explicitly that the particle current $\mathbf{j}$ is zero. *Hint: To do this, you will have to understand the derivation of ch. 8 of the lecture notes and how this changes when the gas is diatomic. You may assume $\mathcal{Q}_{\alpha\beta} = \mathbf{F} = 0$.*

Solution :

(a) Under steady state conditions, the solution to the Boltzmann equation is $f = f^0 + \delta f$, where f^0 is the equilibrium distribution and

$$\delta f = -\frac{\tau f^0}{k_B T} \cdot \frac{\varepsilon - c_p T}{T} \mathbf{v} \cdot \nabla T \quad .$$

For the monatomic ideal gas, $c_p = \frac{5}{2}k_B$. The particle current is

$$\begin{aligned} j^\alpha &= \int d^3p \, v^\alpha \delta f \\ &= -\frac{\tau}{k_B T^2} \int d^3p \, f^0(\mathbf{p}) v^\alpha v^\beta \left(\varepsilon - \frac{5}{2}k_B T \right) \frac{\partial T}{\partial x^\beta} \\ &= -\frac{2n\tau}{3mk_B T^2} \frac{\partial T}{\partial x^\alpha} \langle \varepsilon \left(\varepsilon - \frac{5}{2}k_B T \right) \rangle \quad , \end{aligned}$$

where the average over momentum/velocity is converted into an average over the energy distribution,

$$\tilde{P}(\varepsilon) = 4\pi v^2 \frac{dv}{d\varepsilon} P_M(v) = \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \varepsilon^{1/2} \phi(\varepsilon) e^{-\varepsilon/k_B T} \quad .$$

As discussed in the lecture notes, the average of a homogeneous function of ε under this distribution is given by

$$\langle \varepsilon^\alpha \rangle = \frac{2}{\sqrt{\pi}} \Gamma\left(\alpha + \frac{3}{2}\right) (k_B T)^\alpha \quad .$$

Thus,

$$\langle \varepsilon \left(\varepsilon - \frac{5}{2}k_B T \right) \rangle = \frac{2}{\sqrt{\pi}} (k_B T)^2 \left\{ \Gamma\left(\frac{7}{2}\right) - \frac{5}{2} \Gamma\left(\frac{5}{2}\right) \right\} = 0 \quad .$$

(b) Now we must compute

$$\begin{aligned} j_{\varepsilon^2}^\alpha &= \int d^3p \, v^\alpha \varepsilon^2 \delta f \\ &= -\frac{2n\tau}{3mk_B T^2} \frac{\partial T}{\partial x^\alpha} \langle \varepsilon^3 \left(\varepsilon - \frac{5}{2}k_B T \right) \rangle \quad . \end{aligned}$$

We then have

$$\langle \varepsilon^3 (\varepsilon - \frac{5}{2} k_B T) \rangle = \frac{2}{\sqrt{\pi}} (k_B T)^4 \left\{ \Gamma\left(\frac{11}{2}\right) - \frac{5}{2} \Gamma\left(\frac{9}{2}\right) \right\} = \frac{105}{2} (k_B T)^4$$

and so

$$\mathbf{j}_{\varepsilon^2} = -\frac{35 n \tau k_B}{m} (k_B T)^2 \nabla T \quad .$$

(c) For diatomic gases in the presence of a temperature gradient, the solution to the linearized Boltzmann equation in the relaxation time approximation is

$$\delta f = -\frac{\tau f^0}{k_B T} \cdot \frac{\varepsilon(I) - c_p T}{T} \mathbf{v} \cdot \nabla T \quad ,$$

where

$$\varepsilon(I) = \varepsilon_{\text{tr}} + \varepsilon_{\text{rot}} = \frac{1}{2} m \mathbf{v}^2 + \frac{\mathbf{L}_1^2 + \mathbf{L}_2^2}{2I} \quad ,$$

where $\mathbf{L}_{1,2}$ are components of the angular momentum about the instantaneous body-fixed axes, with $I \equiv I_1 = I_2 \gg I_3$. We assume the rotations about axes 1 and 2 are effectively classical, so equipartition gives $\langle \varepsilon_{\text{rot}} \rangle = 2 \times \frac{1}{2} k_B = k_B$. We still have $\langle \varepsilon_{\text{tr}} \rangle = \frac{3}{2} k_B$. Now in the derivation of the factor $\varepsilon(\varepsilon - c_p T)$ above, the first factor of ε came from the $v^\alpha v^\beta$ term, so this is translational kinetic energy. Therefore, with $c_p = \frac{7}{2} k_B$ now, we must compute

$$\langle \varepsilon_{\text{tr}} (\varepsilon_{\text{tr}} + \varepsilon_{\text{rot}} - \frac{7}{2} k_B T) \rangle = \langle \varepsilon_{\text{tr}} (\varepsilon_{\text{tr}} - \frac{5}{2} k_B T) \rangle = 0 \quad .$$

So again the particle current vanishes.

Note added :

It is interesting to note that there is no particle current flowing in response to a temperature gradient when τ is energy-independent. This is a consequence of the fact that the pressure gradient ∇p vanishes. Newton's Second Law for the fluid says that $n m \dot{\mathbf{V}} + \nabla p = 0$, to lowest relevant order. With $\nabla p \neq 0$, the fluid will accelerate. In a pipe, for example, eventually a steady state is reached where the flow is determined by the fluid viscosity, which is one of the terms we just dropped. (This is called *Poiseuille flow*.) When p is constant, the local equilibrium distribution is

$$f^0(\mathbf{r}, \mathbf{p}) = \frac{p/k_B T}{(2\pi m k_B T)^{3/2}} e^{-p^2/2mk_B T} \quad ,$$

where $T = T(\mathbf{r})$. We then have

$$f(\mathbf{r}, \mathbf{p}) = f^0(\mathbf{r} - \mathbf{v}\tau, \mathbf{p}) \quad ,$$

which says that no new collisions happen for a time τ after a given particle thermalizes. *I.e.* we evolve the streaming terms for a time τ . Expanding, we have

$$\begin{aligned} f &= f^0 - \frac{\tau \mathbf{p}}{m} \cdot \frac{\partial f^0}{\partial \mathbf{r}} + \dots \\ &= \left\{ 1 - \frac{\tau}{2k_B T^2} (\varepsilon(\mathbf{p}) - \frac{5}{2} k_B T) \frac{\mathbf{p}}{m} \cdot \nabla T + \dots \right\} f^0(\mathbf{r}, \mathbf{p}) \quad , \end{aligned}$$

which leads to $\mathbf{j} = 0$, assuming the relaxation time τ is energy-independent.

When the flow takes place in a restricted geometry, a dimensionless figure of merit known as the *Knudsen number*, $\text{Kn} = \ell/L$, where ℓ is the mean free path and L is the characteristic linear dimension associated with the geometry. For $\text{Kn} \ll 1$, our Boltzmann transport calculations of quantities like κ , η , and ζ hold, and we may apply the Navier-Stokes equations¹. In the opposite limit $\text{Kn} \gg 1$, the boundary conditions on the distribution are crucial. Consider, for example, the case $\ell = \infty$. Suppose we have ideal gas flow in a cylinder whose symmetry axis is $\hat{x}$.

¹These equations may need to be supplemented by certain conditions which apply in the vicinity of solid boundaries.

Particles with $v_x > 0$ enter from the left, and particles with $v_x < 0$ enter from the right. Their respective velocity distributions are

$$P_j(\mathbf{v}) = n_j \left(\frac{m}{2\pi k_B T_j} \right)^{3/2} e^{-m\mathbf{v}^2/2k_B T_j} \quad ,$$

where $j = \text{L or R}$. The average current is then

$$\begin{aligned} j_x &= \int d^3v \left\{ n_L v_x P_L(\mathbf{v}) \Theta(v_x) + n_R v_x P_R(\mathbf{v}) \Theta(-v_x) \right\} \\ &= n_L \sqrt{\frac{2k_B T_L}{m}} - n_R \sqrt{\frac{2k_B T_R}{m}} \quad . \end{aligned}$$

(8.2) Consider a classical gas of charged particles in the presence of a magnetic field $\mathbf{B}$. The Boltzmann equation is then given by

$$\frac{\varepsilon - h}{k_B T^2} f^0 \mathbf{v} \cdot \nabla T - \frac{e}{mc} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} = \left(\frac{df}{dt} \right)_{\text{coll}} .$$

Consider the case where $T = T(x)$ and $\mathbf{B} = B \hat{z}$. Making the relaxation time approximation, show that a solution to the above equation exists in the form $\delta f = \mathbf{v} \cdot \mathbf{A}(\varepsilon)$, where $\mathbf{A}(\varepsilon)$ is a vector-valued function of $\varepsilon(\mathbf{v}) = \frac{1}{2} m v^2$ which lies in the (x, y) plane. Find the energy current $\mathbf{j}_\varepsilon$. Interpret your result physically.

Solution: We'll use index notation and the Einstein summation convention for ease of presentation. Recall that the curl is given by $(\mathbf{A} \times \mathbf{B})_\mu = \epsilon_{\mu\nu\lambda} A_\nu B_\lambda$. We write $\delta f = v_\mu A_\mu(\varepsilon)$, and compute

$$\begin{aligned} \frac{\partial \delta f}{\partial v_\lambda} &= A_\lambda + v_\alpha \frac{\partial A_\alpha}{\partial v_\lambda} \\ &= A_\lambda + m v_\lambda v_\alpha \frac{\partial A_\alpha}{\partial \varepsilon} . \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} &= \epsilon_{\mu\nu\lambda} v_\mu B_\nu \frac{\partial \delta f}{\partial v_\lambda} \\ &= \epsilon_{\mu\nu\lambda} v_\mu B_\nu \left(A_\lambda + m v_\lambda v_\alpha \frac{\partial A_\alpha}{\partial \varepsilon} \right) \\ &= \epsilon_{\mu\nu\lambda} v_\mu B_\nu A_\lambda . \end{aligned}$$

We then have

$$\frac{\varepsilon - h}{k_B T^2} f^0 v_\mu \partial_\mu T = \frac{e}{mc} \epsilon_{\mu\nu\lambda} v_\mu B_\nu A_\lambda - \frac{v_\mu A_\mu}{\tau} .$$

Since this must be true for all $\mathbf{v}$, we have

$$A_\mu - \frac{e B \tau}{mc} \epsilon_{\mu\nu\lambda} n_\nu A_\lambda = -\frac{(\varepsilon - h) \tau}{k_B T^2} f^0 \partial_\mu T ,$$

where $\mathbf{B} \equiv B \hat{n}$. It is conventional to define the cyclotron frequency, $\omega_c = eB/mc$, in which case

$$(\delta_{\mu\nu} + \omega_c \tau \epsilon_{\mu\nu\lambda} n_\lambda) A_\nu = X_\mu ,$$

where $\mathbf{X} = -(\varepsilon - h) \tau f^0 \nabla T / k_B T^2$. So we must invert the matrix

$$M_{\mu\nu} = \delta_{\mu\nu} + \omega_c \tau \epsilon_{\mu\nu\lambda} n_\lambda .$$

To do so, we make the *Ansatz*,

$$M_{\nu\sigma}^{-1} = A \delta_{\nu\sigma} + B n_\nu n_\sigma + C \epsilon_{\nu\sigma\rho} n_\rho ,$$

and we determine the constants A , B , and C by demanding

$$\begin{aligned} M_{\mu\nu} M_{\nu\sigma}^{-1} &= (\delta_{\mu\nu} + \omega_c \tau \epsilon_{\mu\nu\lambda} n_\lambda) (A \delta_{\nu\sigma} + B n_\nu n_\sigma + C \epsilon_{\nu\sigma\rho} n_\rho) \\ &= (A - C \omega_c \tau) \delta_{\mu\sigma} + (B + C \omega_c \tau) n_\mu n_\sigma + (C + A \omega_c \tau) \epsilon_{\mu\sigma\rho} n_\rho \equiv \delta_{\mu\sigma} . \end{aligned}$$

Here we have used the result

$$\epsilon_{\mu\nu\lambda} \epsilon_{\nu\sigma\rho} = \epsilon_{\nu\lambda\mu} \epsilon_{\nu\sigma\rho} = \delta_{\lambda\sigma} \delta_{\mu\rho} - \delta_{\lambda\rho} \delta_{\mu\sigma} ,$$

as well as the fact that $\hat{n}$ is a unit vector: $n_\mu n_\mu = 1$. We can now read off the results:

$$A - C \omega_c \tau = 1 , \quad B + C \omega_c \tau = 0 , \quad C + A \omega_c \tau = 0 ,$$

which entail

$$A = \frac{1}{1 + \omega_c^2 \tau^2} \quad , \quad B = \frac{\omega_c^2 \tau^2}{1 + \omega_c^2 \tau^2} \quad , \quad C = -\frac{\omega_c \tau}{1 + \omega_c^2 \tau^2} \quad .$$

So we can now write

$$A_\mu = M_{\mu\nu}^{-1} X_\nu = \frac{\delta_{\mu\nu} + \omega_c^2 \tau^2 n_\mu n_\nu - \omega_c \tau \epsilon_{\mu\nu\lambda} n_\lambda}{1 + \omega_c^2 \tau^2} X_\nu \quad .$$

The α -component of the energy current is

$$j_\epsilon^\alpha = \int \frac{d^3p}{h^3} v_\alpha v_\mu \epsilon A_\mu(\epsilon) = \frac{2}{3m} \int \frac{d^3p}{h^3} \epsilon^2 A_\alpha(\epsilon) \quad ,$$

where we have replaced $v_\alpha v_\mu \rightarrow \frac{2}{3m} \epsilon \delta_{\alpha\mu}$. Next, we use

$$\frac{2}{3m} \int \frac{d^3p}{h^3} \epsilon^2 X_\nu = -\frac{5\tau}{3m} k_B^2 T \frac{\partial T}{\partial x_\nu} \quad ,$$

hence

$$\mathbf{j}_\epsilon = -\frac{5\tau}{3m} \frac{k_B^2 T}{1 + \omega_c^2 \tau^2} \left(\nabla T + \omega_c^2 \tau^2 \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla T) + \omega_c \tau \hat{\mathbf{n}} \times \nabla T \right) \quad .$$

We are given that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and $\nabla T = T'(x) \hat{\mathbf{x}}$. We see that the energy current $\mathbf{j}_\epsilon$ is flowing both along $-\hat{\mathbf{x}}$ and along $-\hat{\mathbf{y}}$. Why does heat flow along $\hat{\mathbf{y}}$? It is because the particles are charged, and as they individually flow along $-\hat{\mathbf{x}}$, there is a Lorentz force in the $-\hat{\mathbf{y}}$ direction, so the energy flows along $-\hat{\mathbf{y}}$ as well.

(8.3) Consider one dimensional motion according to the equation

$$\dot{p} + \gamma p = \eta(t) \quad ,$$

and compute the average $\langle p^4(t) \rangle$. You should assume that

$$\langle \eta(s_1) \eta(s_2) \eta(s_3) \eta(s_4) \rangle = \phi(s_1 - s_2) \phi(s_3 - s_4) + \phi(s_1 - s_3) \phi(s_2 - s_4) + \phi(s_1 - s_4) \phi(s_2 - s_3)$$

where $\phi(s) = \Gamma \delta(s)$. You may further assume that $p(0) = 0$.

Solution :

Integrating the Langevin equation, we have

$$p(t) = \int_0^t dt_1 e^{-\gamma(t-t_1)} \eta(t_1) \quad .$$

Raising this to the fourth power and taking the average, we have

$$\begin{aligned} \langle p^4(t) \rangle &= \int_0^t dt_1 e^{-\gamma(t-t_1)} \int_0^t dt_2 e^{-\gamma(t-t_2)} \int_0^t dt_3 e^{-\gamma(t-t_3)} \int_0^t dt_4 e^{-\gamma(t-t_4)} \langle \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) \rangle \\ &= 3\Gamma^2 \int_0^t dt_1 e^{-2\gamma(t-t_1)} \int_0^t dt_2 e^{-2\gamma(t-t_2)} = \frac{3\Gamma^2}{4\gamma^2} (1 - e^{-2\gamma t})^2 \quad . \end{aligned}$$

We have here used the fact that the three contributions to the average of the product of the four η 's each contribute the same amount to $\langle p^4(t) \rangle$. Recall $\Gamma = 2M\gamma k_B T$, where M is the mass of the particle. Note that

$$\langle p^4(t) \rangle = 3 \langle p^2(t) \rangle^2 \quad .$$

(8.4) A photon gas in equilibrium is described by the distribution function

$$f^0(\mathbf{p}) = \frac{2}{e^{cp/k_B T} - 1} \quad ,$$

where the factor of 2 comes from summing over the two independent polarization states.

- (a) Consider a photon gas (in three dimensions) slightly out of equilibrium, but in steady state under the influence of a temperature gradient ∇T . Write $f = f^0 + \delta f$ and write the Boltzmann equation in the relaxation time approximation. Remember that $\varepsilon(\mathbf{p}) = cp$ and $\mathbf{v} = \frac{\partial \varepsilon}{\partial \mathbf{p}} = c\hat{\mathbf{p}}$, so the speed is always c .
- (b) What is the formal expression for the energy current, expressed as an integral of something times the distribution f ?
- (c) Compute the thermal conductivity κ . It is OK for your expression to involve *dimensionless* integrals.

Solution :

(a) We have

$$df^0 = -\frac{2cp e^{\beta cp}}{(e^{\beta cp} - 1)^2} d\beta = \frac{2cp e^{\beta cp}}{(e^{\beta cp} - 1)^2} \frac{dT}{k_B T^2} \quad .$$

The steady state Boltzmann equation is $\mathbf{v} \cdot \frac{\partial f^0}{\partial \mathbf{r}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$, hence with $\mathbf{v} = c\hat{\mathbf{p}}$,

$$\frac{2c^2 e^{cp/k_B T}}{(e^{cp/k_B T} - 1)^2} \frac{1}{k_B T^2} \mathbf{p} \cdot \nabla T = -\frac{\delta f}{\tau} \quad .$$

(b) The energy current is given by

$$\mathbf{j}_\varepsilon(\mathbf{r}) = \int \frac{d^3p}{h^3} c^2 \mathbf{p} f(\mathbf{p}, \mathbf{r}) \quad .$$

(c) Integrating, we find

$$\begin{aligned} \kappa &= \frac{2c^4 \tau}{3h^3 k_B T^2} \int d^3p \frac{p^2 e^{cp/k_B T}}{(e^{cp/k_B T} - 1)^2} \\ &= \frac{8\pi k_B \tau}{3c} \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty ds \frac{s^4 e^s}{(e^s - 1)^2} \\ &= \frac{4k_B \tau}{3\pi^2 c} \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty ds \frac{s^3}{e^s - 1} \quad , \end{aligned}$$

where we simplified the integrand somewhat using integration by parts. The integral may be computed in closed form:

$$\mathcal{I}_n = \int_0^\infty ds \frac{s^n}{e^s - 1} = \Gamma(n+1) \zeta(n+1) \quad \Rightarrow \quad \mathcal{I}_3 = \frac{\pi^4}{15} \quad ,$$

and therefore

$$\kappa = \frac{\pi^2 k_B \tau}{45 c} \left(\frac{k_B T}{hc} \right)^3 \quad .$$

(8.5) Suppose the relaxation time is energy-dependent, with $\tau(\varepsilon) = \tau_0 e^{-\varepsilon/\varepsilon_0}$. Compute the particle current $\mathbf{j}$ and energy current $\mathbf{j}_\varepsilon$ flowing in response to a temperature gradient ∇T .

Solution :

Now we must compute

$$\begin{aligned} \begin{Bmatrix} j^\alpha \\ j_\varepsilon^\alpha \end{Bmatrix} &= \int d^3p \begin{Bmatrix} v^\alpha \\ \varepsilon v^\alpha \end{Bmatrix} \delta f \\ &= -\frac{2n}{3mk_B T^2} \frac{\partial T}{\partial x^\alpha} \langle \tau(\varepsilon) \begin{Bmatrix} \varepsilon \\ \varepsilon^2 \end{Bmatrix} (\varepsilon - \frac{5}{2}k_B T) \rangle \quad , \end{aligned}$$

where $\tau(\varepsilon) = \tau_0 e^{-\varepsilon/\varepsilon_0}$. We find

$$\begin{aligned} \langle e^{-\varepsilon/\varepsilon_0} \varepsilon^\alpha \rangle &= \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \int_0^\infty d\varepsilon \varepsilon^{\alpha+1/2} e^{-\varepsilon/k_B T} e^{-\varepsilon/\varepsilon_0} \\ &= \frac{2}{\sqrt{\pi}} \Gamma(\alpha + \frac{3}{2}) (k_B T)^\alpha \left(\frac{\varepsilon_0}{\varepsilon_0 + k_B T} \right)^{\alpha + \frac{3}{2}} . \end{aligned}$$

Therefore,

$$\begin{aligned} \langle e^{-\varepsilon/\varepsilon_0} \varepsilon \rangle &= \frac{3}{2} k_B T \left(\frac{\varepsilon_0}{\varepsilon_0 + k_B T} \right)^{5/2} \\ \langle e^{-\varepsilon/\varepsilon_0} \varepsilon^2 \rangle &= \frac{15}{4} (k_B T)^2 \left(\frac{\varepsilon_0}{\varepsilon_0 + k_B T} \right)^{7/2} \\ \langle e^{-\varepsilon/\varepsilon_0} \varepsilon^3 \rangle &= \frac{105}{8} (k_B T)^3 \left(\frac{\varepsilon_0}{\varepsilon_0 + k_B T} \right)^{9/2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{j} &= \frac{5n\tau_0 k_B^2 T}{2m} \frac{\varepsilon_0^{5/2}}{(\varepsilon_0 + k_B T)^{7/2}} \nabla T \\ \mathbf{j}_\varepsilon &= -\frac{5n\tau_0 k_B^2 T}{4m} \left(\frac{\varepsilon_0}{\varepsilon_0 + k_B T} \right)^{7/2} \left(\frac{2\varepsilon_0 - 5k_B T}{\varepsilon_0 + k_B T} \right) \nabla T \quad . \end{aligned}$$

The previous results are obtained by setting $\varepsilon_0 = \infty$ and $\tau_0 = 1/\sqrt{2} n \bar{v} \sigma$. Note the strange result that κ becomes negative for $k_B T > \frac{2}{5} \varepsilon_0$.

(8.6) Use the linearized Boltzmann equation to compute the bulk viscosity ζ of an ideal gas.

- (a) Consider first the case of a monatomic ideal gas. Show that $\zeta = 0$ within this approximation. Will your result change if the scattering time is energy-dependent?
- (b) Compute ζ for a diatomic ideal gas.

Solution :

According to the lecture notes, the solution to the linearized Boltzmann equation in the relaxation time approximation is

$$\delta f = -\frac{\tau f^0}{k_B T} \left\{ m v^\alpha v^\beta \frac{\partial V_\alpha}{\partial x^\beta} - (\varepsilon_{\text{tr}} + \varepsilon_{\text{rot}}) \frac{k_B}{c_V} \nabla \cdot \mathbf{V} \right\} .$$

We also have

$$\text{Tr } \Pi = nm \langle v^2 \rangle = 2n \langle \varepsilon_{\text{tr}} \rangle = 3p - 3\zeta \nabla \cdot \mathbf{V} .$$

We then compute $\text{Tr } \Pi$:

$$\begin{aligned} \text{Tr } \Pi &= 2n \langle \varepsilon_{\text{tr}} \rangle = 3p - 3\zeta \nabla \cdot \mathbf{V} \\ &= 2n \int d\Gamma (f^0 + \delta f) \varepsilon_{\text{tr}} \end{aligned}$$

The f^0 term yields a contribution $3nk_B T = 3p$ in all cases, which agrees with the first term on the RHS of the equation for $\text{Tr } \Pi$. Therefore

$$\zeta \nabla \cdot \mathbf{V} = -\frac{2}{3}n \int d\Gamma \delta f \varepsilon_{\text{tr}} .$$

(a) For the monatomic gas, $\Gamma = \{p_x, p_y, p_z\}$. We then have

$$\begin{aligned} \zeta \nabla \cdot \mathbf{V} &= \frac{2n\tau}{3k_B T} \int d^3p f^0(\mathbf{p}) \varepsilon \left\{ m v^\alpha v^\beta \frac{\partial V_\alpha}{\partial x^\beta} - \frac{\varepsilon}{c_V/k_B} \nabla \cdot \mathbf{V} \right\} \\ &= \frac{2n\tau}{3k_B T} \left\langle \left(\frac{2}{3} - \frac{k_B}{c_V} \right) \varepsilon \right\rangle \nabla \cdot \mathbf{V} = 0 . \end{aligned}$$

Here we have replaced $m v^\alpha v^\beta \rightarrow \frac{1}{3} m v^2 = \frac{2}{3} \varepsilon_{\text{tr}}$ under the integral. If the scattering time is energy dependent, then we put $\tau(\varepsilon)$ inside the energy integral when computing the average, but this does not affect the final result: $\zeta = 0$.

(b) Now we must include the rotational kinetic energy in the expression for δf , and we have $c_V = \frac{5}{2}k_B$. Thus,

$$\begin{aligned} \zeta \nabla \cdot \mathbf{V} &= \frac{2n\tau}{3k_B T} \int d\Gamma f^0(\Gamma) \varepsilon_{\text{tr}} \left\{ m v^\alpha v^\beta \frac{\partial V_\alpha}{\partial x^\beta} - (\varepsilon_{\text{tr}} + \varepsilon_{\text{rot}}) \frac{k_B}{c_V} \nabla \cdot \mathbf{V} \right\} \\ &= \frac{2n\tau}{3k_B T} \left\langle \frac{2}{3} \varepsilon_{\text{tr}}^2 - \frac{k_B}{c_V} (\varepsilon_{\text{tr}} + \varepsilon_{\text{rot}}) \varepsilon_{\text{tr}} \right\rangle \nabla \cdot \mathbf{V} , \end{aligned}$$

and therefore

$$\zeta = \frac{2n\tau}{3k_B T} \left\langle \frac{4}{15} \varepsilon_{\text{tr}}^2 - \frac{2}{5} k_B T \varepsilon_{\text{tr}} \right\rangle = \frac{4}{15} n \tau k_B T .$$

(8.7) Consider a two-dimensional gas of particles with dispersion $\varepsilon(\mathbf{k}) = J\mathbf{k}^2$, where $\mathbf{k}$ is the wavevector. The particles obey photon statistics, so $\mu = 0$ and the equilibrium distribution is given by

$$f^0(\mathbf{k}) = \frac{1}{e^{\varepsilon(\mathbf{k})/k_B T} - 1} \quad .$$

- (a) Writing $f = f^0 + \delta f$, solve for $\delta f(\mathbf{k})$ using the steady state Boltzmann equation in the relaxation time approximation,

$$\mathbf{v} \cdot \frac{\partial f^0}{\partial \mathbf{r}} = -\frac{\delta f}{\tau} \quad .$$

Work to lowest order in ∇T . Remember that $\mathbf{v} = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}}$ is the velocity.

- (b) Show that $\mathbf{j} = -\lambda \nabla T$, and find an expression for λ . Represent any integrals you cannot evaluate as dimensionless expressions.
- (c) Show that $\mathbf{j}_\varepsilon = -\kappa \nabla T$, and find an expression for κ . Represent any integrals you cannot evaluate as dimensionless expressions.

Solution :

(a) We have

$$\begin{aligned} \delta f &= -\tau \mathbf{v} \cdot \frac{\partial f^0}{\partial \mathbf{r}} = -\tau \mathbf{v} \cdot \nabla T \frac{\partial f^0}{\partial T} \\ &= -\frac{2\tau}{\hbar} \frac{J^2 k^2}{k_B T^2} \frac{e^{\varepsilon(\mathbf{k})/k_B T}}{(e^{\varepsilon(\mathbf{k})/k_B T} - 1)^2} \mathbf{k} \cdot \nabla T \end{aligned}$$

(b) The particle current is

$$\begin{aligned} j^\mu &= \frac{2J}{\hbar} \int \frac{d^2 k}{(2\pi)^2} k^\mu \delta f(\mathbf{k}) = -\lambda \frac{\partial T}{\partial x^\mu} \\ &= -\frac{4\tau}{\hbar^2} \frac{J^3}{k_B T^2} \frac{\partial T}{\partial x^\nu} \int \frac{d^2 k}{(2\pi)^2} k^2 k^\mu k^\nu \frac{e^{Jk^2/k_B T}}{(e^{Jk^2/k_B T} - 1)^2} \end{aligned}$$

We may now send $k^\mu k^\nu \rightarrow \frac{1}{2} k^2 \delta^{\mu\nu}$ under the integral. We then read off

$$\begin{aligned} \lambda &= \frac{2\tau}{\hbar^2} \frac{J^3}{k_B T^2} \int \frac{d^2 k}{(2\pi)^2} k^4 \frac{e^{Jk^2/k_B T}}{(e^{Jk^2/k_B T} - 1)^2} \\ &= \frac{\tau k_B^2 T}{\pi \hbar^2} \int_0^\infty ds \frac{s^2 e^s}{(e^s - 1)^2} = \frac{\zeta(2)}{\pi} \frac{\tau k_B^2 T}{\hbar^2} \quad . \end{aligned}$$

Here we have used

$$\int_0^\infty ds \frac{s^\alpha e^s}{(e^s - 1)^2} = \int_0^\infty ds \frac{\alpha s^{\alpha-1}}{e^s - 1} = \Gamma(\alpha + 1) \zeta(\alpha) \quad .$$

(c) The energy current is

$$j_\varepsilon^\mu = \frac{2J}{\hbar} \int \frac{d^2 k}{(2\pi)^2} J k^2 k^\mu \delta f(\mathbf{k}) = -\kappa \frac{\partial T}{\partial x^\mu} \quad .$$

We therefore repeat the calculation from part (c), including an extra factor of Jk^2 inside the integral. Thus,

$$\begin{aligned}\kappa &= \frac{2\tau}{\hbar^2} \frac{J^4}{k_{\text{B}} T^2} \int \frac{d^2 k}{(2\pi)^2} k^6 \frac{e^{Jk^2/k_{\text{B}} T}}{(e^{Jk^2/k_{\text{B}} T} - 1)^2} \\ &= \frac{\tau k_{\text{B}}^3 T^2}{\pi \hbar^2} \int_0^\infty ds \frac{s^3 e^s}{(e^s - 1)^2} = \frac{6 \zeta(3)}{\pi} \frac{\tau k_{\text{B}}^3 T^2}{\hbar^2} \quad .\end{aligned}$$

(8.8) Consider the collisionless Boltzmann equation for the Hamiltonian $\hat{H}(p) = \frac{1}{4}Ap^4$ in one space dimension. Suppose the initial distribution is given by

$$f(x, p, t = 0) = C e^{-x^2/2\sigma^2} e^{-p^2/2\kappa^2} .$$

- (a) Find $f(x, p, t)$ for all $t > 0$.
(b) Find the equation for the locus of points (x, p) for which $f(x, p, t) = \exp(-\alpha^2/2)$.
(c) Express your result in (b) in dimensionless form and plot it for various values of the dimensionless time.

Solution :

(a) The velocity is $v(p) = \partial\varepsilon/\partial p = Ap^3$ and thus

$$f(x, p, t) = f(x - v(p)t, p, 0) = C e^{-(x - Ap^3t)^2/2\sigma^2} e^{-p^2/2\kappa^2} .$$

(b) Clearly we have

$$\frac{(x - Ap^3t)^2}{\sigma^2} + \frac{p^2}{\kappa^2} = \alpha^2 \quad \Rightarrow \quad x(p, t; \alpha) = Ap^3t \pm \sigma \sqrt{\alpha^2 - \frac{p^2}{\kappa^2}} .$$

If we write

$$\bar{x} \equiv \frac{x}{\sigma} , \quad \bar{p} \equiv \frac{p}{\kappa} , \quad s \equiv \frac{A\kappa^3 t}{\sigma} ,$$

then in dimensionless form we have

$$\bar{x}(\bar{p}, s; \alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2} .$$

(c) See fig. 1.

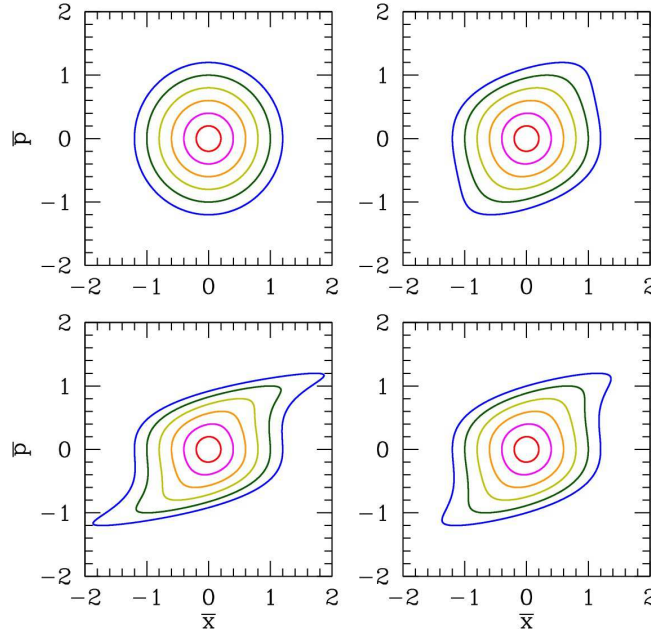


Figure 1: Level sets $\bar{x}(\bar{p}, s; \alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2}$ for $\alpha = 0.2$ (red) to $\alpha = 1.2$ (blue) and $s = 0$ (upper left), $s = \frac{1}{3}$ (upper right), $s = \frac{2}{3}$ (lower right), and $s = 1$ (lower left). Compare with fig. 8.1 of the lecture notes.

(8.9) Consider an ideal gas of point particles in $d = 3$ dimensions with isotropic dispersion $\varepsilon(p) = Ap^\sigma$.

(a) Find the enthalpy per particle $h = \mu + Ts$, where μ is the chemical potential and s is the entropy per particle. (You may find it useful to review some of the material in chapter 4 of the notes.)

(b) Find the thermal conductivity κ within the relaxation time approximation.

Solution :

(a) For the dispersion $\varepsilon(p) = Ap^\sigma$, in d dimensions, one obtains the density of states $g(\varepsilon) = C\varepsilon^{r-1}$ where $r = d/\sigma$ and C is a constant. One then obtains the Helmholtz free energy

$$\begin{aligned} F &= -Nk_B T \log(V/N) - Nk_B T - Nk_B T \log \int_0^\infty d\varepsilon g(\varepsilon) \exp(-\varepsilon/k_B T) \\ &= -Nk_B T \log(V/N) - Nk_B T \log[eA\Gamma(r)] - rNk_B T \log(k_B T) \quad . \end{aligned}$$

We then have

$$\begin{aligned} \mu &= \left(\frac{\partial F}{\partial N} \right)_{T,V} = -k_B T \log(V/N) - k_B T \log[A\Gamma(r)] - rk_B T \log(k_B T) \\ s &= -\frac{1}{N} \left(\frac{\partial F}{\partial T} \right)_{V,N} = k_B \log(V/N) + k_B \log[eA\Gamma(r)] + rk_B \log(k_B T) \quad , \end{aligned}$$

and thus

$$h = \mu + Ts = (r+1)k_B T \quad .$$

The thermal conductivity is obtained from the energy current $\mathbf{j}_\varepsilon = -\kappa \nabla T$, where

$$j_\varepsilon^\alpha = \int d^d p \, \varepsilon v^\alpha \delta f = -\frac{n\tau}{k_B T^2} \langle v^\alpha v^\beta \varepsilon (\varepsilon - h) \rangle \frac{\partial T}{\partial x^\beta} \quad ,$$

where we have used the result (see eqn. 8.71 of the lecture notes)

$$\delta f = -\frac{\tau}{k_B T^2} (\varepsilon - h)(\mathbf{v} \cdot \nabla T) f^0$$

and $h = c_p T$ for our ideal gas. Thus from isotropy we can replace $\langle v^\alpha v^\beta \rangle \rightarrow d^{-1} \mathbf{v}^2 \delta^{\alpha\beta}$, in which case

$$\kappa = \frac{n\tau}{dk_B T^2} \langle \mathbf{v}^2 \varepsilon (\varepsilon - h) \rangle \quad .$$

With $\varepsilon(p) = Ap^\sigma$ we have $\mathbf{v}(p) = \sigma Ap^{\sigma-1} \hat{\mathbf{p}}$ and thus $\mathbf{v}^2 = \sigma^2 A^{2/\sigma} \varepsilon^{2-2\sigma^{-1}}$. We then have

$$\kappa = \frac{n\tau}{dk_B T^2} \sigma^2 A^{2/\sigma} \langle \varepsilon^{3-2\sigma^{-1}} (\varepsilon - (r+1)k_B T) \rangle$$

Now with the density of states $g(\varepsilon) \propto \varepsilon^{r-1}$ we have

$$\langle \varepsilon^s \rangle = \int_0^\infty d\varepsilon \varepsilon^{r-1+s} e^{-\beta\varepsilon} \Big/ \int_0^\infty d\varepsilon \varepsilon^{r-1} e^{-\beta\varepsilon} = \frac{\Gamma(r+s)}{\Gamma(r)} (k_B T)^s \quad .$$

Working out the remaining details, we arrive at the expression

$$\kappa = \frac{2\sigma(\sigma-1)\Gamma(r+3-2\sigma^{-1})}{d\Gamma(r)} A^{2/\sigma} \cdot n\tau k_B \cdot (k_B T)^{2(1-\sigma^{-1})} \quad .$$

As a check, if we set $\sigma = 2$ and $A = 1/2m$ (ballistic dispersion), then for $d = 3$ we have $r = \frac{3}{2}$ and find $\kappa = \frac{5}{2} n\tau k_B^2 T$, which is identical to the result in §8.4.4 of the lecture notes.