PHYSICS 140A : STATISTICAL PHYSICS
MIDTERM EXAMINATION SOLUTIONS

(1) A particle has a g_0-fold degenerate ground state with energy $\varepsilon_0 = 0$ and a g_1-fold degenerate excited state with energy $\varepsilon_1 = \Delta$. A collection of N such particles is arranged on a lattice. Since each particle occupies a distinct position in space, the particles are regarded as distinguishable.

(a) Find the free energy $F(T, N)$.

(b) Find the entropy $S(T, N)$. Sketch $S(T, N)$ *versus* T for fixed N, taking care to evaluate the limiting values $S(T = 0, N)$ and $S(T = \infty, N)$.

Suppose now that the ground state is magnetic, such that in an external field H, the g_0 ground state energy levels are split into $g_0/2$ levels with energy $\varepsilon_{0,+} = +\mu_0 H$ and $g_0/2$ levels with energy $\varepsilon_{0,-} = -\mu_0 H$. (We take g_0 to be even in this case.) The states with energy $\varepsilon_1 = \Delta$ remain g_1-fold degenerate.

(c) Find $F(T, N, H)$.

(d) Find the sero field magnetic susceptibility,

$$\chi(T) = \frac{1}{N} \left(\frac{\partial M}{\partial H} \right)_{H=0} ,$$

where M is the magnetization.

Solution :

(a) We have

$$Z(T, N) = \left(g_0 + g_1 \, e^{-\beta\Delta} \right)^N ,$$

hence

$$\begin{aligned} F(T, N) &= -k_B T \ln Z \\ &= -N k_B T \ln\left(g_0 + g_1 \, e^{-\Delta/k_B T} \right) . \end{aligned}$$

(b) The entropy is

$$S = -\left(\frac{\partial F}{\partial T} \right)_N = N k_B \left[\frac{\Delta}{k_B T} \cdot \frac{g_1 \, e^{-\Delta/k_B T}}{g_0 + g_1 \, e^{-\Delta/k_B T}} + \ln\left(g_0 + g_1 \, e^{-\Delta/k_B T} \right) \right] .$$

From this we see that $S(T = 0, N) = N k_B \ln g_0$ and $S(T = \infty, N) = N k_B \ln(g_0 + g_1)$, which makes physical sense. The sketch should show a smooth interpolation between these values as a function of T.

(c) Now we have

$$F(T,N,H) = -Nk_{\mathrm{B}}T\ln\Big(g_0\cosh(\mu_0 H/k_{\mathrm{B}}T) + g_1\,e^{-\Delta/k_{\mathrm{B}}T}\Big)\ .$$

(d) The magnetization is

$$M(T,H) = -\bigg(\frac{\partial F}{\partial H}\bigg)_{T,N} = \frac{N\mu_0\,g_0\sinh(\mu_0 H/k_{\mathrm{B}}T)}{g_0\cosh(\mu_0 H/k_{\mathrm{B}}T) + g_1\,e^{-\Delta/k_{\mathrm{B}}T}}\ .$$

Taking the derivative and setting $H=0$, we have

$$\chi(T) = \frac{1}{N}\bigg(\frac{\partial M}{\partial H}\bigg)_{T,N,H=0} = \frac{\mu_0^2}{k_{\mathrm{B}}T}\cdot\frac{g_0}{g_0 + g_1\,e^{-\Delta/k_{\mathrm{B}}T}}\ .$$