PHYSICS 140A : STATISTICAL PHYSICS
HW #9 SOLUTIONS

(1) Consider a three-dimensional gas of noninteracting quantum particles with dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{3/2}$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$.

(b) Find expressions for $n(T,z)$ and $p(T,z)$, each expressed as power series in the fugacity z, for both Bose-Einstein and Fermi-Dirac statistics.

(c) Find the virial expansion for the equation of state up to terms of order n^3, for both bosons and fermions.

Solution :

(a) The density of states for dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{\sigma}$ is

$$
\begin{aligned}
g(\varepsilon) &= \mathsf{g}\int \frac{d^d\!k}{(2\pi)^d}\,\delta\big(\varepsilon - Ak^\sigma\big) \\
&= \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\int\limits_0^\infty \! dk\,k^{d-1}\,\frac{\delta\big(k-(\varepsilon/A)^{1/\sigma}\big)}{\sigma A k^{\sigma-1}} = D\,\varepsilon^{\frac{d}{\sigma}-1}\ ,
\end{aligned}
$$

with

$$
D = \frac{2\mathsf{g}}{(2\sqrt{\pi})^d\,\sigma\Gamma(d/2)}\,A^{-d/\sigma}\ .
$$

(b) We have

$$
\begin{aligned}
n(T,z) &= \sum_{j=1}^\infty (\pm 1)^{j-1}\,C_j(T)\,z^j \\
p(T,z) &= k_{\mathrm{B}}T\sum_{j=1}^\infty (\pm 1)^{j-1}\,z^j\,j^{-1}\,C_j(T)\,z^j\ ,
\end{aligned}
$$

where

$$
C_j(T) = \int\limits_{-\infty}^\infty \! d\varepsilon\,g(\varepsilon)\,e^{-j\varepsilon/k_{\mathrm{B}}T} = D\,\Gamma(d/\sigma)\bigg(\frac{k_{\mathrm{B}}T}{j}\bigg)^{d/\sigma}\ .
$$

Thus, we have

$$
\begin{aligned}
\pm\,n v_T &= \sum_{j=1}^\infty j^{-r}\,(\pm z)^j \\
\pm\,p v_T/k_{\mathrm{B}}T &= \sum_{j=1}^\infty j^{-(r+1)}\,(\pm z)^j\ ,
\end{aligned}
$$

where $r = d/\sigma$ and

$$v_T = \frac{1}{D\,\Gamma(d/\sigma)\,(k_B T)^{d/\sigma}} = \frac{(2\sqrt{\pi})^d \sigma\,\Gamma(d/2)}{2\mathrm{g}\,D\,\Gamma(d/\sigma)} \left(\frac{A}{k_B T}\right)^{d/\sigma} .$$

has dimensions of volume.

(c) We now let $x = \pm z$, and interrogate `Mathematica`:

```
In[1]=  y = InverseSeries[x + x^2/2^r + x^3/3^r + x^4/4^r + O[x]^5]

In[2]=  w = y + y^2/2^(r+1) + y^3/3^(r+1) + y^4/4^(r+1) + O[y]^5 .
```

The result is

$$p = nk_B T\Big[1 + B_2(T)\,n + B_3(T)\,n^2 + \ldots\Big] ,$$

where

$$B_2(T) = \mp 2^{-(r+1)}\,v_T$$

$$B_3(T) = \Big(2^{-2r} - 2 \cdot 3^{-(r+1)}\Big) v_T^2$$

$$B_4(T) = \pm 2^{-(3r+1)}\,3^{1-r}\Big(2^{2r+1} - 5 \cdot 3^{r-1} - 2^{r-1}\,3^r\Big) v_T^3 .$$

Substitute $\sigma = \frac{3}{2}$ to find the solution for the conditions given.

(2) You know that at most one fermion may occupy any given single-particle state. A *parafermion* is a particle for which the maximum occupancy of any given single-particle state is k, where k is an integer greater than zero. (For $k = 1$, parafermions are regular everyday fermions; for $k = \infty$, parafermions are regular everyday bosons.) Consider a system with one single-particle level whose energy is ε, *i.e.* the Hamiltonian is simply $\mathcal{H} = \varepsilon n$, where n is the particle number.

(a) Compute the partition function $\Xi(\mu, T)$ in the grand canonical ensemble for parafermions.

(b) Compute the occupation function $n(\mu, T)$. What is n when $\mu = -\infty$? When $\mu = \varepsilon$? When $\mu = +\infty$? Does this make sense? Show that $n(\mu, T)$ reduces to the Fermi and Bose distributions in the appropriate limits.

(c) Sketch $n(\mu, T)$ as a function of μ for both $T = 0$ and $T > 0$.

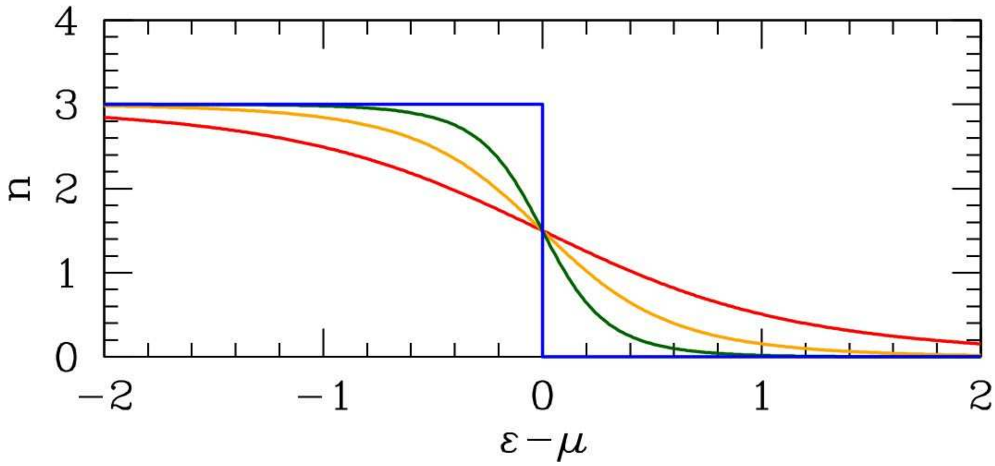

Figure 1: $k = 3$ parafermion occupation number *versus* $\varepsilon - \mu$ for $k_B T = 0$, $k_B T = 0.25$, $k_B T = 0.5$, and $k_B T = 1$. (Problem 2b)

Solution:

The general expression for Ξ is

$$\Xi = \prod_\alpha \sum_{n_\alpha} \left(z\, e^{-\beta\varepsilon_\alpha}\right)^{n_\alpha} .$$

Now the sum on n runs from 0 to k, and

$$\sum_{n=0}^{k} x^n = \frac{1 - x^{k+1}}{1 - x} .$$

(a) Thus,

$$\Xi = \frac{1 - e^{(k+1)\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}} .$$

(b) We then have

$$\begin{aligned} n &= -\frac{\partial\Omega}{\partial\mu} = \frac{1}{\beta}\frac{\partial \ln \Xi}{\partial\mu} \\ &= \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} - \frac{k+1}{e^{(k+1)\beta(\varepsilon-\mu)} - 1} \end{aligned}$$

(c) A plot of $n(\varepsilon, T, \mu)$ for $k = 3$ is shown in Fig. 1. Qualitatively the shape is that of the Fermi function $f(\varepsilon - \mu)$. At $T = 0$, the occupation function is $n(\varepsilon, T = 0, \mu) = k\,\Theta(\mu - \varepsilon)$. This step function smooths out for T finite.

(d) For each $k < \infty$, the occupation number $n(z,T)$ is a finite order polynomial in z, and hence an analytic function of z. Therefore, there is no possibility for Bose condensation except for $k = \infty$.

(3) A gas of quantum particles with photon statistics has dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^4$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$.

(b) Repeat the arguments of §5.5.2 in the Lecture Notes for this dispersion.

(c) Assuming our known values for the surface temperature of the sun, the radius of the earth-sun orbit, and the radius of the earth, what would you expect the surface temperature of the earth to be if the sun radiated particles with this dispersion instead of photons?

Solution :

(a) See the solution to part (a) of problem 1 above. For $d = 3$ and $\sigma = 4$ we have

$$g(\varepsilon) = \frac{\mathsf{g}}{2\pi^2}\,A^{-3/4}\,\varepsilon^{-1/4} \ .$$

(b) Scaling volume by λ scales the lengths by $\lambda^{1/3}$, the quantized wavevectors by $\lambda^{-1/3}$, and the energy eigenvalues by $\lambda^{-4/3}$, since $\varepsilon \propto k^4$. Thus,

$$p = -\left(\frac{\partial E}{\partial V}\right)_S = \frac{4E}{3V} \ ,$$

which says

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p = \tfrac{3}{4}\,p \quad \Rightarrow \quad p(T) = B\,T^{7/4} \ .$$

Indeed,

$$\begin{aligned} p(T) &= -k_{\rm B}T \int\limits_{-\infty}^{\infty} g(\varepsilon)\,\ln\!\big(1 - e^{-\varepsilon/k_{\rm B}T}\big) \\ &= -\frac{\mathsf{g}}{2\pi^2\,A^{3/4}}\,(k_{\rm B}T)^{7/4} \int\limits_{-\infty}^{\infty} du\, u^{-1/4}\,\ln(1 - e^{-u}) \ . \end{aligned}$$

(c) See §5.5.5 of the Lecture Notes. Assume a dispersion of the form $\varepsilon(k)$ for the (non-conserved) bosons. Then the energy current incident on a differential area dA of surface normal to $\hat{\boldsymbol{z}}$ is

$$dP = dA \cdot \int \frac{d^3\!k}{(2\pi)^3}\,\Theta(\cos\theta)\cdot\varepsilon(k)\cdot\frac{1}{\hbar}\frac{\partial\varepsilon(k)}{\partial k_z}\cdot\frac{1}{e^{\varepsilon(k)/k_{\rm B}T} - 1} \ .$$

Note that

$$\frac{\partial\varepsilon(k)}{\partial k_z} = \frac{k_z}{k}\,\frac{\partial\varepsilon}{\partial k} = \cos\theta\,\varepsilon'(k)\ .$$

Now let us assume a power law dispersion $\varepsilon(k) = Ak^\alpha$. Changing variables to $t = Ak^\alpha/k_{\rm B}T$, we find

$$\frac{dP}{dA} = \sigma\,T^{2+\frac{2}{\alpha}}\ ,$$

where

$$\sigma = \zeta\big(2+\tfrac{2}{\alpha}\big)\,\Gamma\big(2+\tfrac{2}{\alpha}\big)\cdot\frac{\mathsf{g}\,k_{\rm B}^{2+\frac{2}{\alpha}}\,A^{-\frac{2}{\alpha}}}{8\pi^2\hbar}\ .$$

One can check that for $\mathsf{g}=2$, $A=\hbar c$, and $\alpha=1$ that this result reduces to Stefan's Law.

Equating the power incident on the earth to that radiated by the earth,

$$4\pi R_\odot^2\cdot\sigma T_\odot^{2(1+\alpha^{-1})}\cdot\frac{\pi R_{\rm e}^2}{4\pi a_{\rm e}^2} = 4\pi R_{\rm e}^2\cdot\sigma T_{\rm e}^{2(1+\alpha^{-1})}\ ,$$

which yields

$$T_{\rm e} = \left(\frac{R_\odot}{2a_{\rm e}}\right)^{\frac{\alpha}{\alpha+1}} T_\odot\ .$$

Plugging in the appropriate constants and setting $\alpha=4$, we obtain $T_{\rm e} = 45.2\,{\rm K}$. Brrr!