PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #1

(1) The *information entropy* of a distribution $\{p_n\}$ is defined as $S = -\sum_n p_n \log_2 p_n$, where n ranges over all possible configurations of a given physical system and p_n is the probability of the state $|n\rangle$. If there are Ω possible states and each state is equally likely, then $S = \log_2 \Omega$, which is the usual dimensionless entropy in units of $\ln 2$.

Consider a normal deck of 52 distinct playing cards. A new deck always is prepared in the same order (A♠ 2♠ $\cdots$ K♣).

(a) What is the information entropy of the distribution of new decks?

(b) What is the information entropy of a distribution of completely randomized decks?

Now consider what it means to shuffle the cards. In an ideal *riffle shuffle*, the deck is split and divided into two equal halves of 26 cards each. One then chooses at random whether to take a card from either half, until one runs through all the cards and a new order is established (see figure).

Figure 1: The riffle shuffle.

(c) What is the increase in information entropy for a distribution of new decks that each have been shuffled once?

(d) Assuming each subsequent shuffle results in the same entropy increase (*i.e.* neglecting redundancies), how many shuffles are necessary in order to completely randomize a deck?

(e) If in parts (b), (c), and (d), you were to use Stirling's approximation,

$$K! \sim K^K\, e^{-K} \sqrt{2\pi K}\ ,$$

how would your answers have differed?

(2) In problem #1, we ran across Stirling's approximation,

$$\ln K! \sim K \ln K - K + \tfrac{1}{2}\ln(2\pi K) + \mathcal{O}\big(K^{-1}\big) \ ,$$

for large K. In this exercise, you will derive this expansion.

(a) Start by writing

$$K! = \int\limits_0^\infty dx \, x^K \, e^{-x} \ ,$$

and define $x \equiv K(t+1)$ so that $K! = K^{K+1}\, e^{-K}\, F(K)$, where

$$F(K) = \int\limits_{-1}^\infty dt \, e^{K f(t)} \ .$$

Find the function $f(t)$.

(b) Expand $f(t) = \sum_{n=0}^\infty f_n \, t^n$ in a Taylor series and find a general formula for the expansion coefficients f_n. In particular, show that $f_0 = f_1 = 0$ and that $f_2 = -\frac{1}{2}$.

(c) If one ignores all the terms but the lowest order (quadratic) in the expansion of $f(t)$, show that

$$\int\limits_{-1}^\infty dt \, e^{-Kt^2/2} = \sqrt{\frac{2\pi}{K}} - R(K) \ ,$$

and show that the remainder $R(K) > 0$ is bounded from above by a function which decreases faster than any polynomial in $1/K$.

(d) *For the brave only!* – Find the $\mathcal{O}\big(K^{-1}\big)$ term in the expansion for $\ln K!$.

(3) A six-sided die is loaded so that the probability to throw a three is twice that of throwing a two, and the probability of throwing a four is twice that of throwing a five.

(a) Find the distribution $\{p_n\}$ consistent with maximum entropy, given these constraints.

(b) Assuming the maximum entropy distribution, given two such identical dice, what is the probability to roll a total of seven if both are thrown simultaneously?

(4) The probability density for a random variable x is given by the Lorentzian,

$$P(x) = \frac{\gamma}{\pi} \cdot \frac{1}{x^2 + \gamma^2} \ .$$

Consider the sum $X_N = \sum_{i=1}^N x_i$, where each x_i is independently distributed according to $P(x_i)$. Find the probability $\mathit{\Pi}_N(Y)$ that $|X_N| < Y$, where $Y > 0$ is arbitrary.