PHYSICS 140A : STATISTICAL PHYSICS
FINAL EXAMINATION

Instructions: *Do problem 4 (34 points) and any two of problems 1, 2, and 3 (33 points each)*

(1) A noninteracting system consists of N dimers. Each dimer consists of two spins, S and σ, where $S \in \{-1, 0, +1\}$ and $\sigma \in \{-1, +1\}$. The Hamiltonian is

$$\hat{H} = -J \sum_{i=1}^{N} S_i \, \sigma_i - \mu_0 H \sum_{i=1}^{N} S_i \, .$$

Thus, the individual dimer Hamiltonian is $h = -JS\sigma - \mu_0 HS$.

(a) Find the N-dimer free energy $F(T, N)$.

(b) Find the average $\langle S \rangle$ and the zero field susceptibility $\chi_S(T) = \frac{\partial \langle S \rangle}{\partial H}\Big|_{H=0}$.

(c) Find the average $\langle \sigma \rangle$ and the zero field susceptibility $\chi_\sigma(T) = \frac{\partial \langle \sigma \rangle}{\partial H}\Big|_{H=0}$.

(d) Examine the $J \to 0$ limits of $\chi_S(T)$ and $\chi_\sigma(T)$ and interpret your results physically.

(2) Recall that a van der Waals gas obeys the equation of state

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT \, ,$$

where v is the molar volume. We showed that the energy per mole of such a gas is given by

$$\varepsilon(T, v) = \tfrac{1}{2} f RT - \frac{a}{v} \, ,$$

where T is temperature and f is the number of degrees of freedom per particle.

(a) For an ideal gas, the adiabatic equation of state is $v\,T^{f/2} = \text{const}$. Find the adiabatic equation of state (at fixed particle number) for the van der Waals gas.

(b) One mole of a van der Waals gas is used as the working substance in a Carnot engine (see Fig. 1). Find the molar volume at v_C in terms of v_B , T_1 , T_2 , and constants.

(c) Find the heat Q_AB absorbed by the gas from the upper reservoir.

(d) Find the work done per cycle, W_cyc. *Hint: you only need to know Q_AB and the cycle efficiency η.*

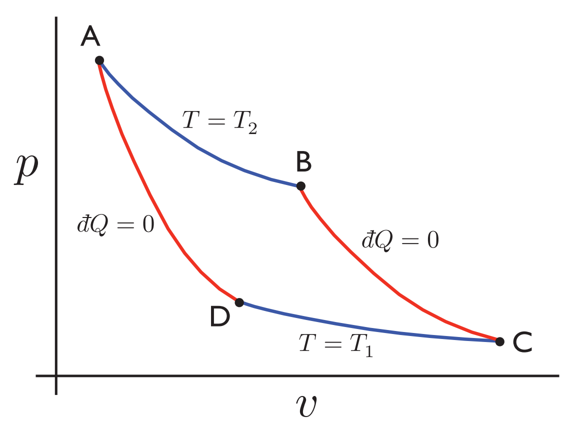

Figure 1: The Carnot cycle.

(3) In homework assignment #9, you showed that the grand partition function for a gas of q-state parafermions is

$$\Xi(T,V,\mu) = \prod_\alpha \left(\frac{1 - e^{(q+1)(\mu-\varepsilon_\alpha)/k_B T}}{1 - e^{(\mu-\varepsilon_\alpha)/k_B T}} \right) ,$$

where the product is over all single particle states. Consider now the case where the number of parafermions is not conserved, hence $\mu = 0$. We call such particles *paraphotons*.

(a) What is the occupancy $n(\varepsilon, T)$ of q-state paraphotons of energy ε?

(b) Suppose the dispersion is the usual $\varepsilon(\boldsymbol{k}) = \hbar c k$. Assuming $\mathsf{g} = 1$, find the single particle density of states $g(\varepsilon)$ in three space dimensions.

(c) Find the pressure $p(T)$. You may find the following useful:

$$\int\limits_0^\infty \! dt \, \frac{t^{r-1}}{e^t - 1} = \Gamma(r)\,\zeta(r) \qquad , \qquad \int\limits_0^\infty \! dt \, t^{r-1} \ln\!\left(\frac{1}{1 - e^{-t}}\right) = \Gamma(r)\,\zeta(r+1) \, .$$

(d) Show that $p = C_q\, n k_B T$, where n is the number density, and C_q is a dimensionless constant which depends only on q.

(4) Provide brief but substantial answers to the following:

(a) A particle in $d = 3$ dimensions has the dispersion $\varepsilon(\boldsymbol{k}) = \varepsilon_0 \exp(ka)$. Find the density of states per unit volume $g(\varepsilon)$. Sketch your result.

(b) Find the information entropy in the distribution $p_n = C\, e^{-\lambda n}$, where $n \in \{0, 1, 2, \ldots\}$. Choose C so as to normalize the distribution.

(c) An ideal gas at temperature $T = 300\,\mathrm{K}$ undergoes an adiabatic free expansion which results in a doubling of its volume. What is the final temperature?

(d) For an N-particle noninteracting system, sketch the contributions ΔC_V to the heat capacity *versus* temperature for (i) a vibrational mode at energy $\hbar\omega_0$, and (ii) a two-level (Schottky) defect with energy splitting $\Delta = \varepsilon_1 - \varepsilon_0$. Take care to identify any relevant characteristic temperatures, as well as the limiting values of ΔC_V.

(5) Write a well-defined expression for the greatest possible number expressible using only five symbols. *Examples:* $1+2+3$, 10^{100}, $\Gamma(99)$. `[50 quatloos extra credit]`