Maxwell equations in media

Maxwell equations in this case can be modified by including to ρ and $\vec{j}$ charges and currents inside atoms. In this case fields (so-called microscopic fields) are changing on microscopic scales comparable with atomic size and number of sources is too large.

It is possible to simplify this picture if we take into account that a detailed behavior of microscopic fields with their drascic changes on atomic scales is not relevant for macroscopic observations. It is sufficient to use macroscopic description of EM waves by using averaged quantities – macroscopic fields and macroscopic sources.

As a result it is possible to write Maxwell equations in a media in the form

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad\qquad \oint_C \vec{H} \cdot d\vec{l} = I + \frac{\partial}{\partial t} \int_S \vec{D} \cdot \vec{n} dA \qquad (1)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{n} dA \qquad (2)$$

$$\vec{\nabla}\vec{D} = \rho \qquad\qquad \oint_S \vec{D} \cdot \vec{n} dA = \int_V \rho d^3 x \qquad (3)$$

$$\vec{\nabla}\vec{B} = 0 \qquad\qquad \oint_S \vec{B} \cdot \vec{n} dA = 0 \qquad (4)$$

Here $\vec{E}$ and $\vec{B}$ are averages of the corresponding microscopic quantities $\vec{E} = \langle \vec{E}_{mic} \rangle$, $\vec{B} = \langle \vec{B}_{mic} \rangle$.

Two new fields $\vec{D}$ (so-called electric displacement) and $\vec{H}$ usually called magnetic field, $\vec{B}$ is now called magnetic inductance. Both $\vec{D}$ and $\vec{H}$ include electric and magnetic moments induced by $\vec{E}$ and $\vec{B}$ fields and averaged over distances large in comparison with molecule (atomic) size.

In a medium $\vec{D}$ and $\vec{H}$ fields are related to $\vec{E}$ and $\vec{B}$ fields through *polarization* and *magnetization*.

Polarization or dipole moment density is defined by :

$$\vec{P}(\vec{x}) = N \langle \vec{p} \rangle \qquad (5)$$

$\vec{p}$ is the dipole moment of a molecule in the medium. Average is taken over a small volume centered at the point $\vec{x}$. N is the number density of molecules at $\vec{x}$.

$\vec{D}$ is related to $\vec{E}$ by the equation

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{6}$$

Magnetization or magnetic moment density is defined by

$$\vec{M}(\vec{x}) = N\langle \vec{m} \rangle \tag{7}$$

$\vec{m}$ is the magnetic moment of a molecule in the medium. Average is taken over a small volume centered at the point $\vec{x}$. N is the number density of molecules at $\vec{x}$.

Magnetization creates an effective current density

$$\vec{j}_M = \vec{\nabla} \times \vec{M} \tag{8}$$

By replacing in the macroscopic Maxwell equations in vacuum $\vec{j}$ by $\vec{j} + \vec{j}_M$ where $\vec{j}$ is macroscopic current density from the flow of free charges in the medium, one can

obtain $\vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \vec{j}_M) \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \vec{\nabla} \times \vec{M}) \Rightarrow \dfrac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{M} = \mu_0 \vec{j}$

Thus $\vec{H} = \dfrac{1}{\mu_0} \vec{B} - \vec{M}$ 　　　(9)

Boundary conditions

If EM fields exist in a region consisting of two different media, the field must satisfy some conditions at the interface surface separating the media (so-called boundary conditions). There are general methods that can be useful for deriving boundary conditions.
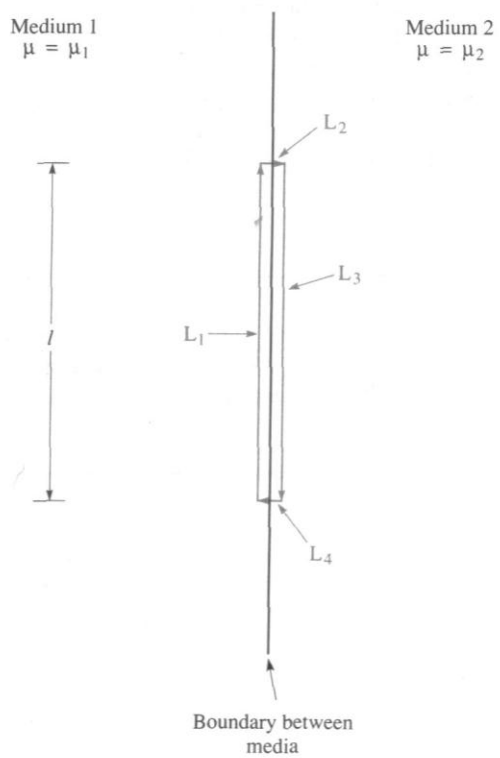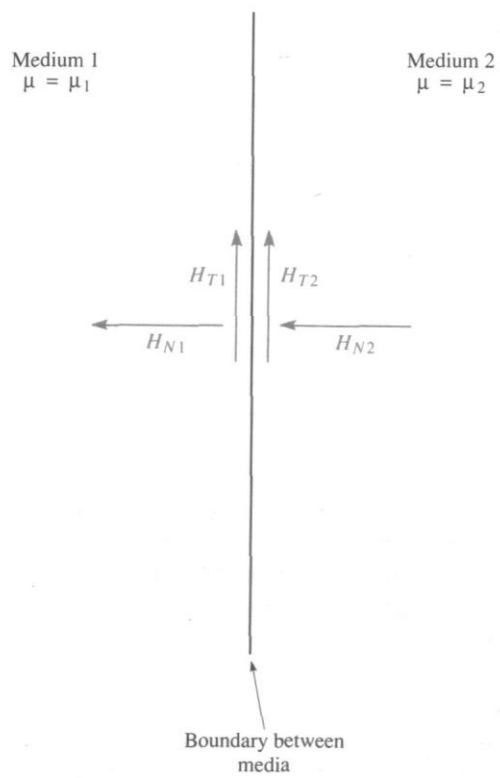
1. **Boundary condition on the tangential component of $\vec{H}$ at a dielectric boundary**

Magnetic field is present at boundary of two materials with different values of ε and μ.

$\vec{H}_1, \vec{H}_2$ can be decomposed into their components tangential to the boundary and normal to the boundary (see Figure). Let us choose the closed path shown in the Figure and consider integral form of Eq. (1):

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{j} \cdot \vec{n} dA + \frac{\partial}{\partial t} \int_S \vec{D} \cdot \vec{n} dA \qquad (1')$$

Since the length of the paths l_2 and l_4 is going to zero, they don't give any contribution in the line integral in L.-H. side of (2'). Contribution to the line integral from l_1 is $T_{t1}l$ and from l_4 is $-T_{t2}l$.

Medium 1
$\mu = \mu_1$

Medium 2
$\mu = \mu_2$

H_{T1} H_{T2}

H_{N1} H_{N2}

Boundary between
media

Medium 1
$\mu = \mu_1$

Medium 2
$\mu = \mu_2$

L_2

L_3

l

L_1

L_4

Boundary between
media

Thus $\oint_C \vec{H} \cdot d\vec{l} = (H_{t1} - H_{t2})l$ (10)

Since the area of the enclosed surface is going to zero, the R.-H. side of equation (10) is going to zero.

As a result, we have from Eqs. (2') and (10)

$H_{t1} = H_{t2}$ (11)

2. **Boundary condition on the tangential component of $\vec{E}$ at the boundary between two media with no charges at the boundary**

In this case the integral form of Eq. (2) will be used.

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{n} dA \qquad (2')$$

We choose the same closed path as shown in the figure. Similarly to (10) we have

$$\oint_C \vec{E} \cdot d\vec{l} = (E_{t1} - E_{t2})l \qquad (12)$$

The R.-H. side of Eq. (1') is equal 0 since the ares of enclosed surface is going to zero.

As a result it follows from (1') and (12), that

$E_{t1} = E_{t2}$ (13)

3. **Boundary condition for good conductors**

a. *Boundary condition for electric field of EM wave.*

Previously you studied that at the surface of a perfect conductor ($\sigma_E \to \infty$) the tangential component of an electric field must be equal 0. For real conductors finite value of E_t will create an extremely large current and redistribute charges in such a way to have $E_t = 0$.

Thus, *an electric field is always perpendicular to the surface of a good conductor.*

b. *Boundary condition for magnetic field of EM wave.*

Using fathor representation, it is possible to write the integral form of Eq. (2) in the following way:

$$\oint_C \vec{E} \cdot d\vec{l} = -i\omega\mu \int_S \vec{H} \cdot \vec{n} dA \qquad (14)$$

Let the closed path to be a small circle on a surface of a good conductor. Then the R.-H. side of Eq. (14) can be approximated by

$$-i\omega\mu \int_S \vec{H} \cdot \vec{n} dA \approx -i\omega\mu H_n A$$

H_n is the normal component of the wave magnetic field and A is the circle area.

The L.-H. side of (14) is equal 0.

Thus, $H_n = 0$. (14')

Magnetic fields of EM wave are approximately tangential to the surface of a good conductor.

4. Relations between normal components at the boundary of two media

a. There is no charges at the boundary surface.

We choose a small cylindrical volume that is located symmetrically relative to the boundary with small faces in both media and with a very small length $\Delta y \to 0$. The Eq. (3):

$$\oint_S \vec{D} \cdot \vec{n} dA = \int_V \rho d^3 x = Q$$ (15)
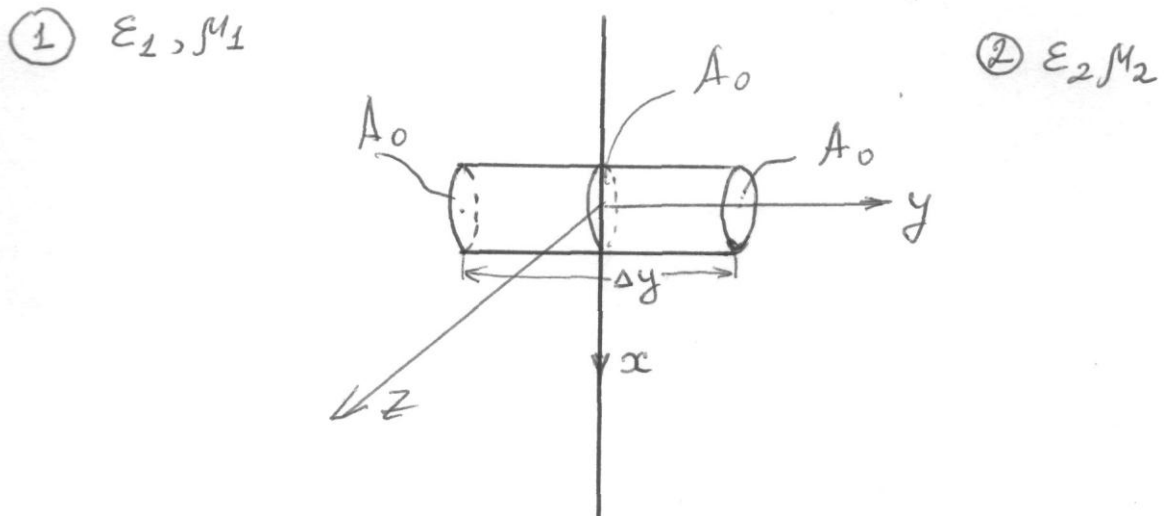
Since $\Delta y \to 0$, $Q \to 0$ as well, so R.-H. side of Eq. (15) is going to 0.

Integral in the L.-H. side of (15) consists of surface integral over cylindrical surface of the cylinder and two intagrals over surface faces. Integral over cylindrical surface is going to 0, since the area of this surface is going to 0 when $\Delta y \rightarrow 0$. Input of integrals over faces in the L.-H. side of (15) can be represented in the following form:

$\vec{D}_2 \cdot \vec{j} A_0 - \vec{D}_1 \cdot \vec{j} A_0$, where $\vec{j}$ is a unit vector in y-direction and A_0 is the area of each face.

As a result we obtain from Eq. (15)

$$D_{2n} = D_{1n} \Rightarrow \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \tag{16}$$

By applying the same approach to Eq. (4)

$$\oint_S \vec{B} \cdot \vec{n} dA = 0$$

We obtain similar relation for normal components of magnetic field at the boundary:

$$B_{2n} = B_{1n} \Rightarrow \mu_2 H_{2n} = \mu_1 H_{1n} \tag{17}$$

b. There is a charge on the boundary surface

We consider Eq. (15) and use the same picture as in case a.

However, in our case when $\Delta y \rightarrow 0$ the R.-H. side of (15)

$$\lim |_{\Delta y \rightarrow 0} \int_V \rho d^3 x = \lim |_{\Delta y \rightarrow 0} [(\rho \Delta y) A_0] = \rho_S A_0 \tag{18}$$

where ρ_s is the surface charge density at the boundary.

As a result, we obtain from (15)

$$\vec{D}_2 \cdot \vec{j} A_0 - \vec{D}_1 \cdot \vec{j} A_0 = \rho_S A_0 \text{ and} \tag{19}$$

$$D_{2n} - D_{1n} = \rho_S \Rightarrow \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_S \tag{20}$$

c. One media is a perfect conductor

In this case $\vec{E}_1 = 0$ and $\vec{D}_1 = 0$ inside the conductor. Then it follows from Eq. (20) that

$$\vec{D}_2 \cdot \vec{n} = \rho_S \Rightarrow D_{2n} = \rho_S \Rightarrow E_{2n} = \rho_S / \varepsilon_2 \tag{21}$$