An unusual approach to Kepler's first law

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Kepler's first law of planetary motion states that the orbits of planets are elliptical, with the sun at one focus. We present an unusual verification of this law for use in classes in mechanics. It has the advantages of resembling the simple verification of circular orbits, and stressing the importance of Kepler's equation.
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I. INTRODUCTION

Kepler's first law of planetary motion states that a point mass moving in a central force field of the form $\mathbf{f}=-(k/r^2)(\mathbf{r}/r)$ will have an orbit which is elliptical in shape with focus at the origin if the motion is bounded. There are many ways to derive this result for students who have mastered only calculus. The purpose of this note is to call attention to a method of showing that elliptical orbits with the proper time dependence satisfy Newton's second law.

This approach does not seem to be used often, but is appealing because it resembles closely the verification of circular orbits. (In this presentation we assume the orbit is elliptical with focus at the center of attraction and that it satisfies Kepler's equation. We then show that this orbit satisfies Newton's second law of motion in the central force field just described. That is, we do not *derive* the elliptical orbits from the assumption of an inverse square force; thus we prefer to describe this note as a "verification" rather than a "derivation.")

The position vector $\mathbf{r}$ describing circular motion (radius a) with uniform angular velocity ω and time t is described by the pair of equations

$$\mathbf{r}=a\cos E\mathbf{i}+a\sin E\mathbf{j}, \tag{1}$$

$$E=\omega t. \tag{2}$$

Motion on an elliptical orbit with eccentricity e and focus at the origin is described by the more complex pair

$$\mathbf{r}=(a\cos E-ae)\mathbf{i}+b\sin E\mathbf{j}, \tag{3}$$

$$E-e\sin E=\omega t. \tag{4}$$

Notice that when $e=0$ (and thus $b=a$), Eqs. (3) and (4) reduce to (1) and (2). The ellipse with position vector $\mathbf{r}=a\cos E\mathbf{i}+b\sin E\mathbf{j}$ would have the origin of coordinates at point C in Fig. 1. Our ellipse (3) has been shifted the distance ae to the left so that the origin is at a focus. Equation (4) is called Kepler's equation.[1] Kepler's equation is a version of Kepler's second law, that the radius vector $\mathbf{r}$ sweeps out equal areas in equal times. (Astronomers call E the eccentric anomaly and ωt the mean anomaly. The term *anomaly* has been used for *angle* by astronomers for hundreds of years because of the irregularities in planetary positions.) A simple geometric derivation of (4) is given in the Appendix to make this paper self-contained. While (1) and (2) can be combined to express the position vector directly in terms of time, $\mathbf{r}=a\cos\omega t\mathbf{i}+a\sin\omega t\mathbf{j}$, this cannot be done in the elliptical case with (3) and (4). We cannot solve (4) for E in terms of t using convenient elementary functions.

II. CIRCULAR ORBITS

We seek orbits of point masses that satisfy Newton's second law of motion in a central force field attracting inversely as the square of the distance:

$$\ddot{\mathbf{r}}=-\frac{k}{r^2}\frac{\mathbf{r}}{r}. \tag{5}$$

Differentiating (1) twice we get

$$\ddot{\mathbf{r}}=-\omega^2(a\cos\omega t\mathbf{i}+a\sin\omega t\mathbf{j})=-\omega^2\mathbf{r}. \tag{6}$$

Comparing this last result with (5) we see that

$$\omega^2=\frac{k}{r^3}. \tag{7}$$

From (7) we conclude that the circle of radius a (1) is a true orbit [satisfies (5)] if the angular velocity ω satisfies $\omega^2=k/a^3$. If T is the period of the orbit, then $\omega=2\pi/T$ and we get from (7) $T^2=(4\pi^2/k)a^3$. This is Kepler's third law.

III. ELLIPTICAL ORBITS

We will now show that the elliptical orbit described by (3) and (4) satisfies Newton's second law (5) in much the same way that the circular orbit given by the pair (1) and (2) does. The calculations are a bit longer, but there are no tricks.

From (3) we see that the length of the position vector $\mathbf{r}$ is given by

$$r^2=(a\cos E-ae)^2+(b\sin E)^2.$$

Using $b^2=a^2-a^2e^2$ we get

$$r^2=(a\cos E-ae)^2+(a^2-a^2e^2)\sin^2 E$$

which simplifies to

$$r=a(1-e\cos E). \tag{8}$$

Now we obtain nice expressions for $\dot{E}$ and $\ddot{E}$. Differentiating Kepler's equation (4) we get

$$\dot{E}(1-e\cos E)=\omega, \tag{9}$$

and using (8) we have

$$\dot{E}=\frac{a\omega}{r}. \tag{10}$$

Differentiating (9) and solving for $\ddot{E}$ we get

$$\ddot{E}=-\frac{e\sin E}{1-e\cos E}(\dot{E})^2.$$

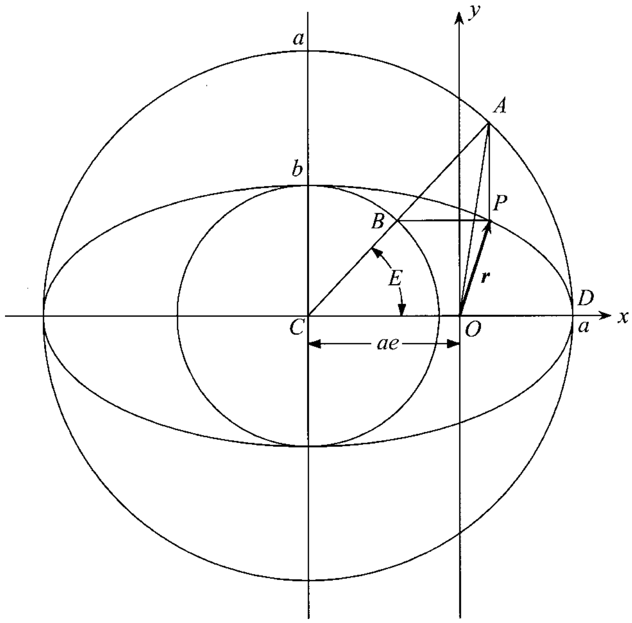

Fig. 1. The elliptical orbit.

Using (8) and (10) we can rewrite this as

$$\ddot{E} = -\frac{a^3\omega^2 e\sin E}{r^3}. \tag{11}$$

Now we can find the acceleration along the elliptical orbit. Differentiating (3) twice we get

$$\dot{\mathbf{r}} = -a\sin E\dot{E}\mathbf{i} + b\cos E\dot{E}\mathbf{j}$$

and

$$\ddot{\mathbf{r}} = (-a\sin E\ddot{E} - a\cos E(\dot{E})^2)\mathbf{i} + (b\cos E\ddot{E} - b\sin E(\dot{E})^2)\mathbf{j}.$$

Next we use (10) and (11) to replace $\dot{E}$ and $\ddot{E}$. We get

$$\ddot{\mathbf{r}} = \left(\frac{a^4\omega^2 e\sin^2 E}{r^3} - \frac{a^3\omega^2\cos E}{r^2}\right)\mathbf{i} - \left(\frac{a^3 b\omega^2 e\sin E\cos E}{r^3} + \frac{a^2 b\omega^2\sin E}{r^2}\right)\mathbf{j}.$$

Factoring out $-a^3\omega^2/r^3$ we have

$$\ddot{\mathbf{r}} = -\frac{a^3\omega^2}{r^3}\left\{(-ae\sin^2 E + r\cos E)\mathbf{i} + \left(be\sin E\cos E + \frac{b}{a}r\sin E\right)\mathbf{j}\right\}.$$

Using (8) to remove r from $\{\cdots\}$ in the above expression we get after simplifying

$$\ddot{\mathbf{r}} = -\frac{a^3\omega^2}{r^3}\{(a\cos E - ae)\mathbf{i} + b\sin E\mathbf{j}\}.$$

Finally, using (3), we have

$$\ddot{\mathbf{r}} = -\frac{a^3\omega^2}{r^3}\mathbf{r}. \tag{12}$$

Notice how (12) for the acceleration on the elliptical path compares with (6), the acceleration on the circular orbit. Substituting (12) into Newton's second law (5) we get

$$-\frac{a^3\omega^2}{r^3}\mathbf{r} = -\frac{k}{r^3}\mathbf{r}.$$

Therefore the elliptical motion described by Eqs. (3) and (4) satisfies Newton's second law if

$$\omega^2 = \frac{k}{a^3}. \tag{13}$$

IV. FINAL REMARKS

(1) One shortcoming of this method is that it is a *verification*, not a *derivation*. We must know relations (3) and (4) (which are mathematical statements of Kepler's first and second laws) before we begin.

(2) Historically, Kepler's laws were known before Newton's laws of motion and gravity. Kepler's first two laws make their initial appearance in his "Astronomia Nova"[2] of 1609. Initially, German astronomers, as well as Galileo, were reluctant to abandon orbits composed of circular motion for Kepler's ellipse. Typical was the reaction of David Fabricius,[3] a clergyman and amateur astronomer who wrote: "With your ellipse you abolish the circularity and uniformity of the motions, which appears to me the more absurd the more profoundly I think about it. ... If you could only preserve the perfect circular orbit, and justify your elliptic orbit by another little epicycle, it would be much better." The first to realize the importance of Kepler's discoveries were the British. In Newton's Principia[4] (1687), he proves that if the orbit of the planet is an ellipse, with one focus at the center of force, then that force must vary inversely as the square of the distance.

(3) Our method emphasizes the importance of Kepler's equation (4). This relation (4) enables us to locate the position of the planet on the elliptical orbit as a function of time. While (4) is always featured in advanced works on celestial mechanics, it seems to be omitted in most mathematical treatments of Kepler's laws in courses in elementary and intermediate mechanics. During the past 300 years, hundreds of papers have been published giving methods of solving (4). The book by Colwell[5] traces this remarkable history.

(4) We recommend Koestler's biography[3] of Kepler for a lively account of his remarkable achievements.

APPENDIX: A DERIVATION OF KEPLER'S EQUATION

We now derive Kepler's equation (4). Our derivation is similar to Moulton's.[1] Refer to Fig. 1. Kepler's second law states that the radius vector $\mathbf{r}$ sweeps out equal areas in equal times as the planet P moves along the ellipse. Let t be the time required for the planet to move from D to P, and let T be the time for a complete traversing of the ellipse. Then we have from Kepler's second law

$$\frac{\text{Area } ODP}{\pi ab} = \frac{t}{T}, \tag{14}$$

where we recall that πab is the area of the full ellipse. Since our ellipse is the result of squashing the large circle of radius a in the vertical direction by the factor b/a, we see that

$$\text{Area } ODP = \frac{b}{a} \text{Area } ODA. \tag{15}$$

Now

$$\text{Area } ODA = \text{Area } CDA - \text{Area } COA$$
$$= \frac{a^2 E}{2} - \frac{(ae)(a \sin E)}{2}. \tag{16}$$

Combining (15) and (16) we see that

$$\text{Area } ODP = \frac{abE}{2} - \frac{eab \sin E}{2}.$$

Substituting this last relation into (14) gives us Kepler's equation

$$E - e \sin E = \frac{2\pi}{T} t = \omega t.$$

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[1]F. R. Moulton, *An Introduction to Celestial Mechanics*, 2nd ed. (Dover, New York, 1970), p. 160.

[2]W. H. Donahue, *Johannes Kepler, New Astronomy* (Cambridge U.P., Cambridge, 1992).

[3]A. Koestler, *The Watershed* (Doubleday, Garden City, NY, 1960), p. 164.

[4]I. Newton, *The Principia*, translated by Andrew Motte (Prometheus Books, Amherst, NY, 1995), pp. 52–53.

[5]P. Colwell, *Solving Kepler's Equation Over Three Centuries* (William-Bell, Richmond, VA, 1993).