A spherical mirror is a reflective segment of a sphere with a radius of curvature $R$. It can be convex (outside surface of a sphere) or concave (inside surface).

First we will consider a concave spherical mirror. The mirror has a radius $R$, and the distance from the mirror to the object is $p$. We will draw the diagram with $p > R$, but our results will hold true for any value of $p$.

Draw two rays from the object to the mirror: one ray passes through the mirror's center of curvature, and therefore is perpendicular to the mirror at the incidence point and returns directly back to the object. The other ray goes to the center of the mirror, and is reflected symmetrically about the horizontal axis. The rays intersect, forming an image a distance $q$ from the mirror. The object has height $h$, the image has height $h'$.

From similar triangles, $h / (p - R) = h' / (R - q)$. Also from similar triangles, $h' / q = h / p$. This gives us

$$\frac{h'}{h} = \frac{R - q}{p - R} = \frac{q}{p}$$

Put this over a common denominator:

$$(R - q)p = q(p - R)$$

$$Rp + Rq = 2qp$$

Divide by $pqR$:

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R}$$
Consider the behavior of this equation if the object is located very far away. Then, all the rays come in parallel to the mirror's axis. The rays are all directed to the point $q$, given by

$$\lim_{p \to \infty} \frac{1}{p} + \frac{1}{q} = \frac{2}{R} \quad \frac{1}{q} = \frac{2}{R} \quad q = \frac{R}{2} \equiv f$$

In terms of the focal length, our equation relating $p, q$ and $f$ is

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

Parallel rays are focused to a distance $f = R / 2$ from the mirror, known as the **focal point**. The ray diagram for parallel rays, as from a source at infinity, looks like this:

![Ray Diagram](image)

In reality, the rays don't all go exactly to the focal point, since the rays that are far away from the axis reflect from points closer to the left than those that come closer to the mirror's center. When an image is formed under some other circumstances, we similarly find that the rays don't all exactly cross. The image produced by a spherical mirror is thus somewhat blurry. This is known as **spherical aberration**. If the mirror is parabolic, rather than spherical, this problem is corrected, as a parabolic mirror has the property that all parallel rays striking it are reflected to exactly the same point. For this reason, mirrors used in optical instruments such as telescopes are parabolic.

But for rays that come in sufficiently close to the mirror's central axis, the performance of a spherical mirror approaches that of a parabolic mirror. Thus, for the purposes of this class, we will use spherical and parabolic mirrors interchangeably.

The quantity $M = -h' / h = -q / p$ is known as the **magnification**. Note that with our sign convention, both $p$ and $q$ in the example above are positive, and $M$ is negative. This means that the image is inverted.
Some examples: Draw ray diagrams, locate the images and determine magnifications for objects located a distance $3f$, $f$ and $0.5f$ away from a concave spherical mirror.

The case for $p = 3f$ is somewhat similar to the one discussed above. Now that we know about the focal point, we will draw different rays: one will reflect through the center of the mirror, as before, but the other will start out parallel to the mirror's axis and be reflected through the focal point, as discussed above:

![Ray Diagram](image)

The image forms at a distance from the mirror $q$, which is given by

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{f}
$$

$$
\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p - f}{pf}
$$

$$
q = \frac{p}{p - f}f = \frac{3f}{3f - f}f = \frac{3}{2}f
$$

Thus, in this case, the distance of the image from the mirror is $3/2 = 1.5$ times the distance of the focal point, as shown in the diagram. The image is in front of the mirror, and rays of light actually cross there, so it is a **real image**.

The magnification is given by

$$
M = -\frac{q}{p} = -\frac{f}{p - f} = -\frac{f}{3f - f} = -\frac{1}{2}
$$

The image is therefore inverted, and half the size of the object. The location and nature of the image and the magnification match up well with our ray diagram; this is a useful check on the calculations.

For the object located at the focal point, $p = f$, the ray diagram looks like this:

![Ray Diagram](image)
The outgoing rays don't appear to cross, and therefore it seems that there is no image. Let us confirm this. For the image location, we have
\[ q = \frac{p}{p - f} \frac{f}{f - f} = \frac{f}{0} \]

So, in fact, the rays never cross. As \( p \) approaches \( f \) from above, \( q \) approaches infinity. Thus, the situation with the object at the focal point is often referred to as an image at infinity. This makes sense given the reversibility of light rays: if parallel rays coming from an object at infinity are focused to an image at the focal point, then we expect rays originating from an object at a focal point to emerge as parallel (“image at infinity”).

The magnification in this case also approaches infinity.

Finally, for the object located at \( p = 0.5f \), the ray diagram looks like this:

The rays diverge in front of the mirror, so there is no real image. However, tracing the emerging rays to their apparent point of origin behind the mirror gives an upright, magnified virtual image (an image constructed by tracing back rays, rather than from an intersection of actual rays). The image position is
\[ q = \frac{p}{p - f} = \frac{1/2f}{1/2f - f} = -f \]
\[ M = \frac{-q}{p} = \frac{-f}{1/2f} = 2 \]

As expected, \( q \) is negative, so the image is on the other side of the mirror (and is virtual). The magnification is positive, meaning that the image is upright. The image is twice as big as the object.
Convex spherical mirrors

It turns out that convex spherical mirrors obey exactly the same equations as concave spherical mirrors, except the radius of curvature is taken to be negative (since the center of curvature is on the opposite side of the mirror). Since \( f = R / 2 \), this means that the focal length is negative (the focal point is also on the opposite side of the mirror).

Instead of repeating all our geometry and showing that it still works the same way, we will simply use this in a couple of examples and show that we obtain acceptable results.

Consider first an object placed a distance \( 2|f| \) away from a convex spherical mirror. The ray diagram looks like this (note the difference that the ray parallel to the mirror's axis is now reflected directly away from the focal point):

From carefully drawing the diagram and tracing back the rays behind the mirror, we expect an upright virtual image, smaller than the object and closer to the surface of the mirror than the focal point. Let's see if our equations predict the same thing. Writing the focal length as \( f = -|f| \) so that it is understood to be negative, we get

\[
\frac{1}{p} + \frac{1}{q} = -\frac{1}{|f|} \quad \frac{1}{q} = -\frac{1}{|f|} - \frac{1}{p} = -\frac{p + |f|}{|f|} \\
q = -\frac{p}{p + |f|}|f| = -\frac{2|f|}{2|f| + |f|}|f| = -\frac{2}{3}|f| \\
M = -\frac{q}{p} = -\frac{(2/3)|f|}{2|f|} = \frac{1}{3}
\]

Thus we indeed have an upright virtual image \( 1/3 \) the size of the object, located a distance of \( 2/3|f| \) behind the surface of the mirror.

Now consider an object at a distance \( 1/2 |f| \) in front of a convex spherical mirror. The diagram now looks like this:
We expect an upright virtual image somewhat smaller than the object, much closer to the mirror than the focal point. Our equations give us

\[ q = -\frac{p}{p + |f|} |f| = -\frac{1/2|f|}{1/2|f| + |f|} |f| = -\frac{1}{3} |f| \]

\[ M = -\frac{q}{p} = -\frac{-1/3|f|}{1/2|f|} = \frac{2}{3} \]

The image is thus behind the mirror (virtual), 3 times closer to the mirror than the focal point, 2/3 the size of the object, and upright.

Notice that unlike for the concave mirror, the character of the image for a convex mirror does not change much depending on the distance of the object. The image always remains upright, virtual, closer to the mirror than the focal point, and smaller than the object. This was not the case for the convex mirror, where the type of image we got depended on whether the object was closer or farther than the focal point.

The flat mirror is a special case of either the convex mirror or the concave mirror, with \( R \) taken to infinity. This gives

\[ \frac{1}{p} + \frac{1}{q} = 0 \quad q = -p \quad M = 1 \]

The image formed by a flat mirror is thus virtual, located at the same distance from the mirror as the object, upright, and the same size as the object, as we expect.
Refraction by a curved surface

When light is incident on a curved surface with radius of curvature $R$, it will be refracted to form an image. The ray diagram looks something like this:

The lower ray from the object is directed towards the center of the sphere; it is perpendicular to the surface where it is incident on it, so it is not refracted and continues on in a straight line. The upper ray strikes the sphere at an angle $q_1$ and is refracted to an angle $q_2$ according to Snell's law, $n_1 \sin q_1 = n_2 \sin q_2$. Some straightforward but extremely tedious geometry and trigonometry will show that the rays will intersect and form an image at a point given by

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

Note that in this case, our convention is that $q$ is positive when the image is on the opposite side from the object, not the same side as for mirrors. This is because with refraction, rays pass through the surface rather than being reflected, so real images form on the opposite side.

A special case is a flat surface, where $R$ is infinite. In this case, the image distance is

$$\frac{n_1}{p} + \frac{n_2}{q} = 0 \quad q = -\frac{n_2}{n_1}p$$

The image is formed on the same side as the object. An example is the image of an object underwater. The ray diagram for this example looks like this:
A virtual image is formed by tracing back the refracted rays. The image is closer to the surface than the object by a factor of \( \frac{n_2}{n_1} = 1.00 / 1.33 = 0.75 \).

**Refraction by a thin lens**

A thin lens consists of a pair of curved surfaces, which change the direction of travel of light based on where on the surface light enters the lens. The lens acts in much the same way as a mirror, except that light passes through the lens rather than being reflected from it. As with the spherical surface, since light moves through the lens, possibly creating a real image on the other side, \( p \) is positive in the direction from which light is incident, while \( q \) is positive in the direction where light emerges. So, if the object and the image are on opposite sides of the lens, \( p \) and \( q \) are both positive.

A perfect lens focuses parallel rays of light towards its focal point (if it is convergent) or directly away from its focal point (if it is divergent), as shown in the following diagrams:

![Convergent lens](image1)

Convergent lens

![Divergent lens](image2)

Divergent lens

If the two sides of the lens have radii of curvature \( R_1 \) and \( R_2 \), the focal length of the lens is determined by the *lens maker's equation*:

\[
\frac{1}{f} = \left( \frac{n_L}{n_M} - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

The focal length is considered positive if the lens is convergent; this means that the rays will be focused to a point on the opposite side of the object. It is negative if the lens is divergent; this means that the rays will diverge from the focal point on the same side as the object. For an ideal thin lens, the focal length describes all its optical properties. Once we have calculated it, we don't need to know anything else about the lens unless we are interested in studying imperfections such as aberration.

**Example:** A lens a convex front side with a radius of curvature of 45 centimeters, and a concave back side with a radius of curvature of 30 centimeters. It is made from a material with an index of refraction of 1.62. What is its focal length? Is it convergent or divergent?
The cross-section of the lens looks like this:

The focal length is

\[
\frac{1}{f} = \left( \frac{1.62}{1.00} - 1 \right) \left( \frac{1}{45\text{cm}} - \frac{1}{30\text{cm}} \right) = -0.00689\text{cm}^{-1}
\]

\[f = -145\text{cm}\]

The lens is therefore divergent, with a focal length of -145 cm. We could have predicted that it would be divergent, since its divergent (back) side has a smaller radius of curvature than the front, and is therefore more curved and stronger than the front.

Here, \(n_L\) is the index of refraction of the lens while \(n_M\) is the index of refraction of the medium in which the lens is to be used (1.00 for air or vacuum). The sign convention on \(R_1\) and \(R_2\) is such that \(R_1\) is the radius of curvature on the side of the incident light, and is positive if the surface is convex (converging), while \(R_2\) is the radius of curvature on the back of the lens, and is positive if the surface is concave (diverging). Note that this means that if both sides are converging, the result will be a lens that is even more converging; if one side is converging and one is diverging, then the terms in the equation will tend to cancel each other out.

The object distance \(p\), image distance \(q\) and focal length \(f\) are related by the same equation as for the mirrors:

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{f}
\]

The only thing that is different are the sign conventions for \(f\) and \(q\), and the fact that the light travels through the lens rather than being reflected from it.

**Example:** Draw a ray diagram for an object located 25 cm in front of a convergent lens with a focal length of 60 cm. Where is the image, and what is the magnification?

We draw two rays, one parallel to the axis of the lens (which will be refracted directly towards the focal point) and one crossing the middle of the lens (which will continue in the same direction, since the two sides of the lens in the center are parallel, and the lens refracts light just as a pane of glass would):
The rays will not cross on the opposite side of the lens, but tracing them back, we obtain a virtual image. Its position is given by

\[ \frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{1}{60\text{cm}} - \frac{1}{25\text{cm}} = -0.0233\text{cm}^{-1} \]

\[ q = -43\text{cm} \]

The negative number for a lens means that the image is virtual, and on the same side as the object. The magnification is

\[ M = \frac{-q}{p} = \frac{-43\text{cm}}{25\text{cm}} = 1.7 \]

The object is therefore magnified by a factor of 1.7. This lens is an example of a magnifying glass.

**Example:** Do the same for a convergent lens with a focal length of 60cm, but with the object located 100cm away.

\[ \frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{1}{60\text{cm}} - \frac{1}{100\text{cm}} = 0.00667\text{cm}^{-1} \]

\[ q = 150\text{cm} \]

\[ M = \frac{-q}{p} = -1.50 \]
The image is real, upside-down, and 1.50 times the size of the object. If a screen was placed at the location of the image, the image would be projected on the screen. This is how a projector or a camera works.

**Example:** An object is placed 60 cm away from a divergent lens with a focal length of -30 cm. Locate the image and find the magnification.

The ray diagram (note that the parallel ray is refracted directly away from the focal point):

![Ray Diagram](image)

The location of the image (remember that the focal length is negative)

\[
\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = -\frac{1}{30\,cm} - \frac{1}{60\,cm} = -0.05\,cm^{-1}
\]

\[q = -20\,cm\]

The magnification is \(M = \frac{-20}{60} = 1/3\). The image is upright, virtual and 1/3 the size of the object.