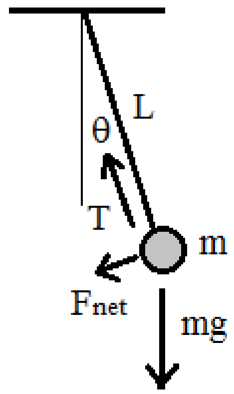
Lecture 2 Notes: 06 / 28

The Simple Pendulum: Force Diagram

A simple pendulum consists of a small mass suspended on an approximately massless, non-stretchable string. It is free to oscillate from side to side. The forces acting on the mass are the force of gravity and the tension in the string:



The tension cancels out the component of *mg* that lies along the string; this keeps the object from accelerating in the direction of the string, and thus keeps the string's length constant. The net force is simply the remaining component of *mg*, which is pointed perpendicular to the string and is equal to:

$$F_{net} = -mgsin(\theta)$$

(We have made the net force negative since it points in the direction of decreasing θ).

Equations of Motion

The mass travels along an arc on a circle; the displacement along the arc is given by $x = L\theta$ and so the acceleration is given by $a = L\alpha$, where $\alpha = d^2\theta/dt^2$. Thus, Newton's Law gives

$$ma = F_{net} \qquad mL\alpha = -mgsin(\theta)$$

$$m\alpha = -\frac{mg}{L}sin(\theta)$$

This is not Hooke's Law, since a sine function appears on the right hand side. However, for small angles, the sine function can be expanded as follows:

$$sin(\theta) = \theta - \frac{1}{6}\theta^3 + ...$$

The angle here must be expressed in radians. If the angle is sufficiently small, then we can just keep the first term in the expansion, and replace $sin(\theta)$ with θ. Our equation of motion then becomes

$$m\alpha = -\frac{mg}{L}\theta$$

This is clearly the equation of motion for a harmonic oscillator, with θ playing the role of x, α taking the role of a, and $k = mg / L$. Thus, for small displacements, the pendulum will oscillate with simple harmonic motion (this is just another example of simple harmonic motion being a nearly universal behavior for systems near equilibrium).

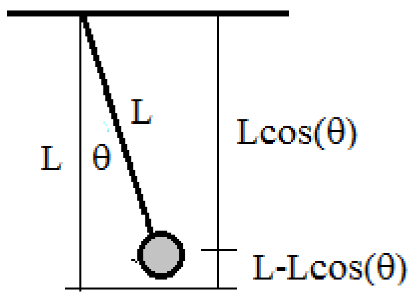
Using the results from the previous lecture, the pendulum's angular frequency, frequency and period are:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}} \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad T = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$$

Note that the pendulum's frequency does not depend on how heavy an object is attached to the string: all masses oscillate with the same frequency. However, the pendulum is sensitive to the length of the string and the acceleration due to gravity.

Energy of the Pendulum

The pendulum only has gravitational potential energy, as gravity is the only force that does any work. Let us define the potential energy as being zero when the pendulum is at the bottom of the swing, $\theta = 0$. When the pendulum is elsewhere, its vertical displacement from the $\theta = 0$ point is $h = L - L\,cos(\theta)$ (see diagram)



The potential energy of the pendulum is therefore

$$U = mgh = mgL(1 - cos(\theta))$$

For small angles, this turns out to correspond to the potential energy of the harmonic oscillator, just as the force does. We can expand the cosine function as follows:

$$cos(\theta) = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + ... \approx 1 - \frac{1}{2}\theta^2$$

This gives the following expression for *U*:

$$U \approx \frac{1}{2}mgL\theta^2$$

The total energy of the pendulum is the potential energy at the maximum displacement, when the displacement is equal to the amplitude:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}mgL\theta^2 = \frac{1}{2}mgLA^2$$

Conservation of energy therefore gives a kinematic equation for the velocity as a function of deflection angle:

$$v^2 = gL\left(A^2 - \theta^2\right)$$

Note that in the above expressions, the amplitude *A* has units of radians, and measures the maximum deflection angle. This is unlike the case of a mass on a spring, where the amplitude has units of distance.

Some problems:

1. If you want a pendulum to have a period of 1 second, how long should the pendulum be?

$$T = 2\pi\sqrt{\frac{L}{g}} = 1s \qquad L = \frac{gT^2}{4\pi^2} = \frac{9.8m/s^2 \times (1s)^2}{4\pi^2} = 0.248m = 24.8cm$$

2. If you take the same pendulum to the Moon, where g = 1.68m/s^2, what will be its period there?

$$T = 2\pi\sqrt{\frac{L}{g_{moon}}} = 2\pi\sqrt{\frac{0.248m}{1.68m/s^2}} = 2.41s$$

Pendulum clocks:

Since the pendulum's period is not very consistent and depends only the pendulum length and the force of gravity, pendulums were historically used in clocks. An arm would swing back and fourth every second, or several seconds. There would be a spring-loaded driving force that would keep the arm going, and a mechanism coupling it to the machinery of the clock.

These devices were reasonably good at keeping time, but sensitive to variations in local gravity. A pendulum clock made for one location would not necessarily keep good time in another.

Problem:

Suppose a pendulum clock is made to run with g = 9.810 m/s^2. What would be the error, in seconds per day, if you moved the clock to a location with g = 9.813 m/s^2?

Since the period is inversely proportional to the square root of g, the new period is

$$T_{new} = T_{old}\sqrt{\frac{g_{old}}{g_{new}}} = T_{old}\sqrt{\frac{9.810}{9.813}} = 0.999847T_{old}$$

The clock would therefore run slightly fast (it would take slightly less than a second to move the second hand once). How many seconds' error would accumulate per day? Well, in a 24-hour period, the number of seconds counted by this clock would be:

$$N = \frac{24hr \times 3600s/hr}{0.9999847s} = 86413$$

However, a day only has 86,400 seconds. Therefore, this clock would run fast by 13 seconds per day in the new location.

Damped Oscillators

Now we will add a particular kind of frictional force to our oscillating systems. This kind of force, called *linear damping*, is proportional to minus the velocity:

$$F_d = -bv$$

The force is directed opposite to the velocity, so it will always slow the object down, as a friction force must.

Linear damping is not necessarily the most common type of friction - we have seen kinetic friction before, which does not depend on the speed (but does depend on the direction of the velocity, being always directed opposite to it). Friction coming from air resistance is typically quadratic.

Linear damping does occur when an object moves through a viscous medium. This is used in shock absorbers. Also, the internal friction in a spring that eventually causes an oscillating mass on a spring to stop, is well approximated by linear damping.

If we do have linear damping, our equation of motion becomes:

$$F_{net} = -kx - bv = ma$$

$$ma + bv + kx = 0$$

We can try an exponential solution to this equation. This represents a mass that begins away from equilibrium, but approaches the equilibrium point at later times, without oscillating at all. This behavior is seen in strongly damped systems. We guess the following solution:

$$x(t) = Ae^{-\gamma t} \quad v(t) = \frac{dx}{dt} = -\gamma Ae^{-\gamma t} \quad a(t) = \frac{dv}{dt} = \gamma^2 Ae^{-\gamma t}$$

Plugging this into our equation, we obtain

$$m\gamma^2 Ae^{-\gamma t} - b\gamma Ae^{-\gamma t} + kAe^{-\gamma t} = 0$$

$$m\gamma^2 - b\gamma + k = 0$$

$$\gamma_\pm = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}$$

Note that this only gives a real number if $b^2 >= 4mk$. So, this is indeed a solution that is only valid for sufficiently high damping; the behavior at lower damping is somewhat different. Oscillators that do satisfy this condition are called ***overdamped oscillators***, which is a strange name, since they don't oscillate at all! Both possible values of γ give valid solutions: in fact, any combination of the two solutions is a valid solution. In general,

$$x(t) = Ae^{-\gamma_+ t} + Be^{-\gamma_- t}$$

The exact values of the amplitudes *A* and *B* depend on the initial conditions, as shown in the following problem:

Problem:

An overdamped oscillator starts at rest with a displacement x_0 from the origin. What is the displacement of this oscillator as a function of time?

We know that the oscillator starts at $x = x_0$ at $t = 0$. Therefore,

$$x(0) = A + B = x_0$$

We have two unknowns, however, A and B, so we need another equation. That is provided by the fact that the oscillator starts at rest:

$$v(0) = 0 = \frac{dx}{dt}(t = 0) = -(\gamma_+ A + \gamma_- B)$$

$$\gamma_+ A + \gamma_- B = 0$$

Using the first equation, $B = x_0 - A$. Plugging this into the second equation,

$$\gamma_+ A + \gamma_- (x_0 - A) = 0$$

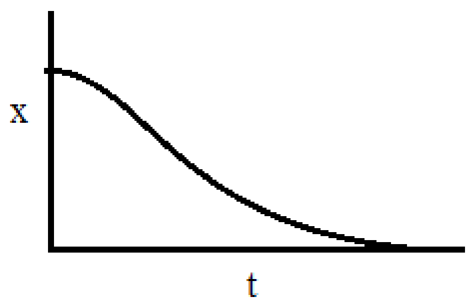
$$A = -\frac{\gamma_- x_0}{\gamma_+ - \gamma_-} = \frac{-b + \sqrt{b^2 - 4mk}}{2\sqrt{b^2 - 4mk}} x_0$$

$$B = x_0 - A = \frac{\gamma_+ x_0}{\gamma_+ - \gamma_-} = \frac{b + \sqrt{b^2 - 4mk}}{2\sqrt{b^2 - 4mk}} x_0$$

With the constants A and B determined, we can now find the displacement as a function of time. It is given by:

$$x(t) = Ae^{-\gamma_+ t} + Be^{-\gamma_- t} \qquad \gamma_\pm = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}$$

For a typical choice of parameters, the graph of $x(t)$ looks something like this:



Underdamped Oscillators

An underdamped oscillator has $b^2 < 4mk$. In that case, we need a somewhat different solution. Unlike overdamped oscillators, underdamped oscillators do oscillate, but the amplitude of the oscillations decreases exponentially. We will try a combination of a cosine and an exponential function as a solution:

$$x(t) = Ae^{-\gamma t} cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -A\gamma\, e^{-\gamma t} cos(\omega t + \phi) - A\omega e^{-\gamma t} sin(\omega t + \phi)$$

$$a(t) = \frac{dv}{dt} = A(\gamma^2 - \omega^2) e^{-\gamma t} cos(\omega t + \phi) + 2A\gamma\omega e^{-\gamma t} sin(\omega t + \phi)$$

This is getting a bit complicated. Inserting this into our equation of motion and grouping terms by sine and cosine, we get:

$$\left(m(\gamma^2 - \omega^2) - b\gamma + k\right) Ae^{-\gamma t} cos(\omega t + \phi) +$$

$$+ (2m\gamma\omega - b\omega)\, Ae^{-\gamma t} sin(\omega t + \phi) = 0$$

In order for this equation to hold at all times, both prefactors must be zero. We thus have a system of equations:

$$m(\gamma^2 - \omega^2) - b\gamma + k = 0$$

$$2m\gamma\omega - b\omega = 0$$

Solve the second equation for γ, and plug into the first:

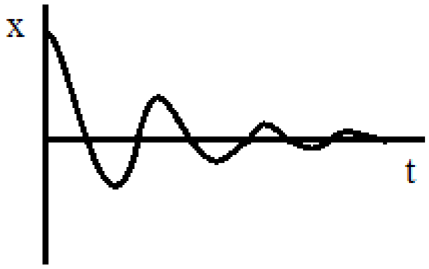
$$\gamma = \frac{b}{2m}$$

$$\frac{b^2}{4m} - m\omega^2 - \frac{b^2}{2m} + k = 0$$

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The angular frequency is shifted from the value for the simple, undamped harmonic oscillator. However, for small damping, the shift is not large. The quantity under the square root is positive for $b^2 < 4mk$, so this solution is indeed valid over the entire region where the overdamped solution is not.

For an underdamped system, the solution looks something like this. The amplitude decays exponentially while the system oscillates about the equilibrium point.



Problem:

A 1kg mass on a spring with k = 40N/m oscillates with an amplitude of 25 cm. A minute later, it is oscillating with an amplitude of 10 cm. What is the damping coefficient?

The amplitude of the oscillations goes as

$$A(t) = A_0 e^{-\gamma t}$$

If the amplitude after a time t is A_1, then we get the following equation:

$$A_1 = A_0 e^{-\gamma t} \qquad -\gamma t = ln\left(\frac{A_1}{A_0}\right)$$

$$\gamma = \frac{1}{t} ln\left(\frac{A_0}{A_1}\right) = \frac{1}{60s} ln\left(\frac{25cm}{10cm}\right) = 0.0153s^{-1}$$

The damping rate is related to the damping coefficient as follows:

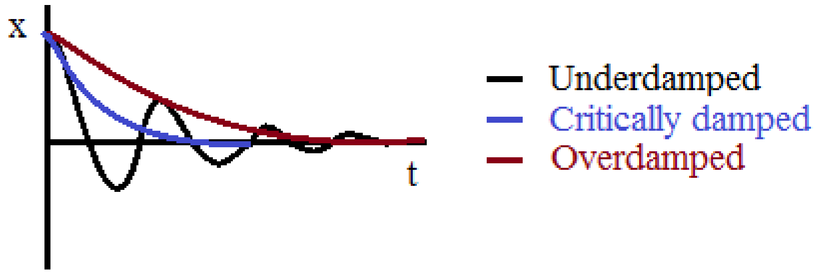
$$\gamma = \frac{b}{2m} \qquad b = 2m\gamma = 2 \times 1kg \times 0.0153s^{-1} = 0.0306kg\ s^{-1}$$

Critical Damping

The point of transition between underdamping and overdamping, where $b^2 = 4mk$, is known as ***critical damping***. The general solution for a critically damped oscillator is

$$x(t) = (A + Bx)e^{-\gamma t} \qquad \gamma = \frac{b}{2m}$$

An oscillator returns to the equilibrium position the fastest if it is critically damped (See figure below). An overdamped oscillator will drift towards equilibrium slowly, held back by excess friction, while an underdamped oscillator will oscillate back and forth across the equilibrium point before settling there. A critically damped oscillator minimizes the time it takes to settle to equilibrium. For this reason, shock absorbers are often designed to be critically damped.



Problem:

A shock absorber has a moving piston with a mass of 5kg, and is critically damped. When an additional 10kg is attached to the shock absorber, the system is undergoes damped oscillations with a frequency of 5 hz. What is the spring constant and the damping coefficient of the shock absorber?

Let $m_0 = 5kg$ be the shock absorber's intrinsic mass, and $m_1 = 15kg$ be the mass of the shock absorber plus the additional weight. We have two unknown quantities: the shock absorber's spring constant and damping coefficient, so we'll need two equations to solve for these quantities. The first equation comes from the condition that the shock absorber is critically damped with no additional mass:

$$b^2 = 4m_0k$$

The second equation comes from the frequency when the additional mass is added:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m_1} - \frac{b^2}{4m_1^2}}$$

Plug the expression for b^2 from the first equation into the second to solve for k, then use k to obtain b:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_1} - \frac{4m_0k}{4m_1^2}} = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}\left(1 - \frac{m_0}{m_1}\right)} = \frac{1}{2\pi}\sqrt{\frac{2}{3}\frac{k}{m_1}}$$

$$k = \frac{3 \times 4\pi^2}{2}m_1f^2 = 6\pi^2 \times 15kg \times (5s^{-1})^2 = 2.2 \times 10^4 kg \cdot s^{-2}$$

$$b = \sqrt{4m_0k} = \sqrt{4 \times 5kg \times 2.2 \times 10^4 kg \cdot s^{-2}} = 670kg \cdot s^{-1}$$