Physics 4E   Quiz 3 Solutions   Spring 2011

Problem 1

$$E_n = \frac{\pi^2\hbar^2 n^2}{2m_e L^2} \Rightarrow L^2 = \frac{\pi^2\hbar^2 n^2}{2m_e E} \Rightarrow L = \pi\sqrt{\frac{\hbar^2}{2m_e E}}\, n$$

Using $\frac{\hbar^2}{2m_e} = 3.81\,eV\mathring{A}^2$, $E = 5eV \Rightarrow L = \pi\sqrt{\frac{3.81}{5}}\,\mathring{A}\cdot n \Rightarrow$

$\boxed{L = 2.74\mathring{A}\cdot n}$ smallest well: $\boxed{L = 2.74\mathring{A}}$

or, for $n = 2, 3$, $L = 5.48\mathring{A}$ or $L = 8.23\mathring{A}$.

(b) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$. By symmetry, $\langle p\rangle = 0$

from $E_1 = \frac{\hbar^2\pi^2}{2m_e L^2} = \frac{\langle p^2\rangle}{2m_e} \Rightarrow \langle p^2\rangle = 2m_e\cdot E_1$

$\Rightarrow \Delta p = \sqrt{2m_e E_1} = \sqrt{2m_e c^2 E_1}\,/c = 2260.5\,eV/c$

$\boxed{\Delta p = 2260.5\,eV/c}$

(c) $\Delta p\cdot L = 2260.5\frac{eV}{c}\cdot 2.74\mathring{A}\cdot\frac{\hbar}{\hbar} = \frac{6193.85\,eV\mathring{A}}{1973\,eV\mathring{A}}\cdot\hbar = 3.14\hbar$

$\Rightarrow \boxed{\Delta p\cdot L = 3.14\hbar}$

Or: $p^2 = \frac{\hbar^2\pi^2}{L^2} \Rightarrow p^2L^2 = \pi^2\hbar^2 \Rightarrow \boxed{\Delta p\cdot L = \pi\hbar}$

(d) Wavefunction is $\psi_1(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi}{L}x$

$$\frac{P(x=\frac{L}{2})}{P(x=L/6)} = \frac{|\psi_1(L/2)|^2}{|\psi_1(L/6)|^2} = \frac{\sin^2\frac{\pi}{L}\frac{L}{2}}{\sin^2\frac{\pi}{L}\frac{L}{6}} = \frac{1^2}{(\frac{1}{2})^2} = \boxed{4}$$

Problem 2

$$\psi(x) = C e^{-x^2/a^2}$$

$$1 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = C^2 \int dx\, e^{-2x^2/a^2} = \sqrt{\frac{\pi a^2}{2}}\, C^2$$

$$\Rightarrow C = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{a^{1/2}} \Rightarrow \boxed{C = 0.893\,\text{Å}^{-1/2}}$$

(b) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, by symmetry $\langle x\rangle = 0$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 = C^2 \int dx\, x^2 e^{-2x^2/a^2} =$$

$$= C^2 \cdot \frac{1}{2\cdot 2} a^2 \cdot \sqrt{\frac{\pi a^2}{2}}$$; using that $C^2\sqrt{\frac{\pi a^2}{2}} = 1 \Rightarrow$

$$\langle x^2\rangle = \frac{a^2}{4} \Rightarrow \boxed{\Delta x = \frac{a}{2} = 0.5\,\text{Å}}$$

(c) From the formula for the harmonic oscillator wavefunctions,

$$\frac{m\omega}{2\hbar} = \frac{1}{a^2} \Rightarrow \omega = \frac{2\hbar}{ma^2}$$ $\Rightarrow$ since this is the ground

state, $$\mathcal{E}_0 = \frac{\hbar\omega}{2} = \frac{2\hbar^2}{2m_e a^2} = \frac{\hbar^2}{m_e a^2} = 2\times\frac{3.81\,\text{eV}\text{Å}^2}{1\,\text{Å}^2} = 7.62\,\text{eV}$$

$$\boxed{\mathcal{E}_0 = 7.62\,\text{eV}}$$

Problem 3

10,000 incident, 100 transmitted $\Rightarrow$ $T = \frac{1}{100} = 0.01 \Rightarrow$

$\boxed{R = 0.99}$ $\Rightarrow$ k_2 is very small $\qquad \boxed{k = \sqrt{\frac{2m}{\hbar^2}(E-V)}}$

$$R = \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \Rightarrow R(k_1+k_2)^2 = (k_1-k_2)^2 \Rightarrow \sqrt{R}\,(k_1+k_2) = k_1-k_2 \Rightarrow$$

$$\Rightarrow (\sqrt{R}+1)k_2 = (1-\sqrt{R})k_1 \Rightarrow k_2^2(1+\sqrt{R})^2 = k_1^2(1-\sqrt{R})^2 \Rightarrow$$

$$\cancel{\frac{2m}{\hbar^2}}(E-1eV)(1+\sqrt{R})^2 = \cancel{\frac{2m}{\hbar^2}}E(1-\sqrt{R})^2 \Rightarrow$$

$$\Rightarrow E\left[(1+\sqrt{R})^2-(1-\sqrt{R})^2\right] = 1eV(1+\sqrt{R})^2 \Rightarrow$$

$$\Rightarrow E\left[\cancel{1}+2\sqrt{R}+\cancel{R}-\cancel{1}+2\sqrt{R}-\cancel{R}\right] = 1eV(1+\sqrt{R})^2 \Rightarrow E\cdot 4\sqrt{R} = 1eV(1+\sqrt{R})^2 \Rightarrow$$

$$\Rightarrow \boxed{E = 1eV\,\frac{(1+\sqrt{R})^2}{4\sqrt{R}}}$$ note that $\frac{(1+\sqrt{R})^2}{4\sqrt{R}} \geq 1$ always

$$\Rightarrow \boxed{E = 1.000006\,eV} \quad (a)$$

For tunneling through the barrier: again $T = \frac{1}{100} = 0.01$

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(V_0-E)}\cdot\Delta x}$$ with $\Delta x = 2\mathring{A}$, $E \simeq 1eV$

$$-2\sqrt{\frac{2m}{\hbar^2}(V_0-E)}\,\Delta x = \ln T \Rightarrow \frac{2m}{\hbar^2}(V_0-E) = \frac{(\ln T)^2}{4(\Delta x)^2} \Rightarrow$$

$$V_0 - E = \frac{\hbar^2}{2m}\frac{(\ln T)^2}{4(\Delta x)^2} \Rightarrow \boxed{V_0 = E + \frac{\hbar^2}{2m}\frac{(\ln T)^2}{4(\Delta x)^2}} \Rightarrow$$

$$V_0 = 1eV + 3.81\,eV\mathring{A}^2\cdot\frac{(\ln 0.01)^2}{4\cdot 4\,\mathring{A}^2} \Rightarrow \boxed{V_0 = 6.05\,eV}$$