1) We consider front propagation for the modified Fisher equation $u_t = u_{xx} + u(1-u^2)$. We want to study the system as a propagating front so we let $u(x,t) = u(x-ct)$ where c is the front propagation speed. Then with $\xi = x - ct$,

$$\frac{d^2u}{d\xi^2} + c\frac{du}{d\xi} + u(1-u^2) = 0 \tag{1}$$

As in the text, we can see that we have the two-dimensional dynamical system

$$\begin{aligned} \frac{du}{d\xi} &= v \\ \frac{dv}{d\xi} &= -u(1-u^2) - cv \end{aligned} \tag{2}$$

Then we have fixed points at $(u^*, v^*) = (0,0)$ and $(u^*, v^*) = (\pm 1, 0)$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ 3u^2 - 1 & -c \end{pmatrix} \tag{3}$$

The trace is always $-c$. For $(u^*, v^*) = (\pm 1, 0)$, the determinant is -2, so we have a saddle point. For $(u^*, v^*) = (0,0)$, the determinant is 1. If $u(x,t)$ describe a density, it must be positive and, as described in the text (at the end of 8.1.3) , we rule out $c < 2$, so the fixed point is a stable node.

For the stability, we write $u(x,t) = U(x-ct) + \delta u(x,t)$, with $U(\xi)$ a solution. Linearizing in δu, we obtain the PDE

$$\frac{\partial \delta u}{\partial t} = \frac{\partial^2 \delta u}{\partial x^2} + (1 - 3U^2)\delta u \tag{4}$$

We shift to a moving frame defined by $\xi = x - ct$ and $s = t$. We then get the equation (using eqs. (8.31) and (8.32) from the notes):

$$\frac{\partial \delta u}{\partial s} = \frac{\partial^2 \delta u}{\partial \xi^2} + (1 - 3U(\xi)^2)\delta u \tag{5}$$

This is a linear and autonomous PDE and solutions can be written in the form $u(\xi, s) = f(\xi)exp(-\lambda s)$, where

$$f'' + cf' + (\lambda + 1 - 3U^2)f = 0 \tag{6}$$

To get rid of f', we right $f(\xi) = \psi(\xi)exp(-\frac{c\xi}{2})$ to obtain $-\frac{d^2\psi}{d\xi^2} + W(\xi)\psi = \lambda\psi$, where $W(\xi) = 3U^2(\xi) + \frac{c^2}{4} - 1$ is the 'potential'. Then if $|c| > 2$ we get all positive eigenvalues and otherwise get negative eigenvalues. Therefore solutions with $|c| < 2$ are unstable.

2 For the predator-prey model given by

$$
\begin{aligned}
u_t &= Du_{xx} - uv \\
v_t &= \lambda D v_{xx} + uv
\end{aligned} \tag{7}
$$

If we examine the possibility of a traveling front solution, with $u(x,t) = u(x-ct)$ and $v(x,t) = v(x-ct)$, we obtain the coupled ode system:

$$
\begin{aligned}
Du'' + cu' - uv &= 0 \\
\lambda D v'' + cv' + uv &= 0
\end{aligned} \tag{8}
$$

We now have a four dimensional system:

$$
\begin{aligned}
\frac{du}{d\xi} &= z \\
D\frac{dz}{d\xi} &= -cz + uv \\
\frac{dv}{d\xi} &= w \\
\lambda D\frac{dw}{d\xi} &= -cw - uv
\end{aligned} \tag{9}
$$

We then get a Jacobian that looks like (in the order u, v, z, w):

$$
J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ v/D & u/D & -c/D & 0 \\ -v/D\lambda & -u/D\lambda & 0 & -c/D\lambda \end{pmatrix} \tag{10}
$$

We observe that fixed points exist at $(0,0,0,0)$, $(m_1,0,0,0)$, and $(0,m_2,0,0)$, with m_1, m_2 arbitrary. First examine the Jacobian evaluated at $(0,0,0,0)$:

$$
J_{(0,0,0,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -c/D & 0 \\ 0 & 0 & 0 & -c/D\lambda \end{pmatrix} \tag{11}
$$

The eigenvalues are 0 (double), $-c/D$, and $-c/DL$. For the other two, we let $m_1 = m_2 = K$, because that's our boundary condition. This gives Jacobians of:

$$
J_{(K,0,0,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & K/D & -c/D & 0 \\ 0 & -K/D\lambda & 0 & -c/D\lambda \end{pmatrix} \tag{12}
$$

and

$$J_{(0,K,0,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K/D & 0 & -c/D & 0 \\ -K/D\lambda & 0 & 0 & -c/D\lambda \end{pmatrix} \tag{13}$$

Using MATLAB, Mathematica, or by hand, we find that the eigenvalues for the $(K, 0, 0, 0)$ are $0, -c/D$ and $\frac{-c\pm\sqrt{c^2+4KD}}{2D}$. For $(0, K, 0, 0)$ we similarly find $0, -c/D\lambda$ and $\frac{-c\pm\sqrt{c^2-4DK\lambda}}{2D\lambda}$.

We have three special cases: where $D \approx 0$, where $\lambda \approx 0$, and where D is very small but λ is very large such that $D\lambda \approx O(1)$. First examine when $D \approx 0$. Then we have a 2D system:

$$\begin{aligned} c\frac{du}{d\xi} &= uv \\ c\frac{dv}{d\xi} &= -uv \end{aligned} \tag{14}$$

with Jacobian

$$\begin{pmatrix} v/c & u/c \\ -v/c & -u/c \end{pmatrix} \tag{15}$$

with eigenvalues 0 and (u - v)/c and fixed points at $(K, 0)$ and $(0, K)$.

For $\lambda \approx 0$, we have a 3D system

$$\begin{aligned} \frac{du}{d\xi} &= z \\ D\frac{dz}{d\xi} &= -cz + uv \\ c\frac{dv}{d\xi} &= -uv \end{aligned} \tag{16}$$

with eigenvalues 0 and $\frac{1}{2cD}(uD + zc^2 \pm \sqrt{u^2D^2 - 2uDzc^2 + c^4z^2 + 4Dvc^2})$ with fixed points at $(0, K, 0)$, $(K, 0, 0)$, and $(0, 0, 0)$.

Finally, we have the system where $\lambda D \approx O(1)$ when $D \approx 0$. This system is

$$\begin{aligned} c\frac{du}{d\xi} &= uv \\ \frac{dv}{d\xi} &= w \\ \lambda D\frac{dw}{d\xi} &= -cw - uv \end{aligned} \tag{17}$$

with all the same fixed points as before.

3 We compute the "growth rate" η for the Brusselator within a purely linearized treatment of the problem. First, compute η at fixed $Q(\epsilon = 0) = a/\sqrt{D_u D_v}$. We have the coupled RDE's:

$$\begin{aligned} u_t = D_u u_{xx} + f(u,v) = D_u u_{xx} + a - (1+b)u + u^2 v \\ v_t = D_v v_{xx} + g(u,v) = D_v v_{xx} + bu - u^2 v \end{aligned} \tag{18}$$

The fixed point occurs at $(u^*, v^*) = (a, b/a)$. Linearizing and Fourier-decomposing, we get

$$J = \begin{pmatrix} f_u - q^2 D_u & f_v \\ g_v & g_u - q^2 D_v \end{pmatrix} = \begin{pmatrix} -(1+b) - 2uv - q^2 D_u & u^2 \\ b - 2uv & u^2 - q^2 D_v \end{pmatrix} = \begin{pmatrix} b - 1 - q^2 D_u & a^2 \\ -b & -a^2 - q^2 D_v \end{pmatrix} \tag{19}$$

Let $q = \pm Q$, so $Q^2 = \frac{D_u g_v + D_v f_u}{2 D_u D_v} = -\frac{D_u a^2 + D_v (b-1)}{2 D_u D_v}$. Additionally, note that the trace and determinant of the above Jacobian are $Tr = b - 1 - a^2 - q^2(D_u + D_v)$ and $D = -bq^2 D_v + a^2 + q^2 D_v + q^2 a^2 D_u + q^4 D_u D_v$, respectively.

Now let's examine the growth rate at a fixed Q. Then $D_u Q^2 = c$, $D_v Q^2 = a/c$, and $(c+1)^2 = b_T$. With $\epsilon = b - b_T$, we can simplify our trace and determinant to $Tr = (1 + \frac{1}{c})(c^2 - a^2) + \epsilon$ and $D = -\frac{\epsilon a^2}{c}$. Then

$$\begin{aligned} \frac{T^2}{4} - D = \frac{((1+1/c)(c^2-a^2))^2}{4} + \frac{\epsilon}{2}(c^2 + c - a^2 - \frac{a^2}{c}) + \epsilon a^2/c + \epsilon^2/4 \\ = (\frac{(1+1/c)(c^2-a^2)}{2})^2 + \frac{\epsilon}{2}(c^2(1+1/c) - a^2(1-1/c)) \\ = (\frac{(1+1/c)(c^2-a^2)}{2})^2 (1 + \frac{\epsilon}{2} \frac{4(c^2(1+1/c) - a^2(1-1/c))}{((1+1/c)(c^2-a^2))^2}) \end{aligned} \tag{20}$$

so

$$\sqrt{\frac{T^2}{4} - D} \approx \frac{(1+1/c)(c^2-a^2)}{2} + \frac{\epsilon}{2} \frac{(c^2(1+1/c) - a^2(1-1/c))}{(1+1/c)(c^2-a^2)} \tag{21}$$

So the (-) eigenvalue gives us

$$-\frac{T}{2} + \sqrt{\frac{T^2}{4} - D} = \frac{\epsilon a^2}{(c+1)(a^2-c^2)} = \frac{a^2(b - b_T)}{(c+1)(a^2-c^2)} \tag{22}$$

which is what the notes have for (9.67).

Now we allow Q to vary. Then

$$Q^2(\epsilon) = \frac{D_u g_v + D_v f_u}{2 D_v D_u} = \frac{-a^2 D_u + D_v(b-1)}{2 D_u D_v} \tag{23}$$

So $D_u Q^2 = c + \epsilon/2$ and $D_v Q^2 = \frac{a^2}{c}(1+\epsilon/2c)$. Then

$$Tr = (1+1/c)(c^2-a^2) + \frac{\epsilon}{2}(1-a^2/c^2) \tag{24}$$

$$D = ((c+1)^2+\epsilon-1-c-\frac{\epsilon}{2})(-a^2-\frac{a^2}{c}(1+\frac{\epsilon}{2c}))+a^2(\epsilon+(c+1)^2) = -\frac{a^2\epsilon}{c}-\frac{a^2\epsilon^2}{4c^2} \tag{25}$$

If this all gets plugged in to determine the eigenvalues, we get

$$-\frac{T}{2}-\sqrt{T/4-D} \approx -\frac{1}{2}(1+1/c)(c^2-a^2)-$$
$$\frac{1}{2}(1+1/c)(a^2-c^2)(1+\frac{\epsilon}{2}\frac{(1-a^2/c^2)(1+1/c)(c^2-a^2)+4a^2/c}{((1+1/c)(c^2-a^2))^2})-\frac{\epsilon}{4}(1-a^2/c^2)$$
$$= \frac{a^2(b-b_T)}{(c+1)(a^2-c^2)} \tag{26}$$

4

We consider the real Ginsburg-Landau equation, $\psi_t = \mu\psi + \psi_{xx} - |\psi|^2\psi$, with ψ a complex field. We investigate the stability of static solutions of the form $\psi(x) = \sqrt{\mu - Q^2}e^{iQx}$. We first write $\psi(x,t) = \sqrt{\mu-Q^2}e^{iQx} + \eta(x)e^{\lambda t} = \xi(x) + \eta(x)e^{\lambda t}$. Then by only keeping terms that are linear in η we get

$$\lambda\eta = \mu\eta + \eta_{xx} - 2|\xi|^2\eta - \xi^2\eta^* \tag{27}$$

We can define η by its expansion

$$\eta = a_0 e^{iQx} + \sum(a_k e^{i(Q+k)x} + b_k e^{i(Q-k)x}) \tag{28}$$

Then

$$\lambda_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} \mu-(Q+k)^2-2(\mu-Q^2) & -(\mu-Q^2) \\ -(\mu-Q^2) & \mu-(Q-k)^2-2(\mu-Q^2) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{29}$$

which gives us $\lambda_{k\pm} = -(\mu-Q^2) - k^2 \pm \sqrt{(2Qk)^2+(\mu-Q^2)^2}$. To make $\xi(x)$ stable, we require the eigenvalue λ_{k+} to be negative, which gives us our Eckhaus instability.