1) We consider front propagation for the modified Fisher equation $u_t = u_{xx} + u(1 - u^2)$. We want to study the system as a propagating front so we let $u(x, t) = u(x - ct)$ where $c$ is the front propagation speed. Then with $\xi = x - ct$,

$$\frac{d^2u}{d\xi^2} + c \frac{du}{d\xi} + u(1 - u^2) = 0$$

(1)

As in the text, we can see that we have the two-dimensional dynamical system

$$\frac{du}{d\xi} = v$$

$$\frac{dv}{d\xi} = -u(1 - u^2) - cv$$

(2)

Then we have fixed points at $(u^*, v^*) = (0, 0)$ and $(u^*, v^*) = (±1, 0)$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ 3u^2 - 1 & -c \end{pmatrix}$$

(3)

The trace is always $-c$. For $(u^*, v^*) = (±1, 0)$, the determinant is -2, so we have a saddle point. For $(u^*, v^*) = (0, 0)$, the determinant is 1. If $u(x, t)$ describe a density, it must be positive and, as described in the text (at the end of 8.1.3), we rule out $c < 2$, so the fixed point is a stable node.

For the stability, we write $u(x, t) = U(x - ct) + \delta u(x, t)$, with $U(\xi)$ a solution. Linearizing in $\delta u$, we obtain the PDE

$$\frac{\partial \delta u}{\partial t} = \frac{\partial^2 \delta u}{\partial x^2} + (1 - 3U^2)\delta u$$

(4)

We shift to a moving frame defined by $\xi = x - ct$ and $s = t$. We then get the equation (using eqs. (8.31) and (8.32) from the notes):

$$\frac{\partial \delta u}{\partial s} = \frac{\partial^2 \delta u}{\partial \xi^2} + (1 - 3U(\xi)^2)\delta u$$

(5)

This is a linear and autonomous PDE and solutions can be written in the form $u(\xi, s) = f(\xi)e^{\lambda(-\lambda s)}$, where

$$f'' + cf' + (\lambda + 1 - 3U^2)f = 0$$

(6)

To get rid of $f'$, we write $f(\xi) = \psi(\xi)e^{(-c\xi/2)}$ to obtain $-\frac{d^2\psi}{d\xi^2} + W(\xi)\psi = \lambda \psi$, where $W(\xi) = 3U^2(\xi) + \frac{c^2}{4} - 1$ is the ‘potential’. Then if $|c| > 2$ we get all positive eigenvalues and otherwise get negative eigenvalues. Therefore solutions with $|c| < 2$ are unstable.

2 For the predator-prey model given by
\[ u_t = Du_{xx} - uv \]
\[ v_t = \lambda Du_{xx} + uv \] (7)

If we examine the possibility of a traveling front solution, with \( u(x,t) = u(x-ct) \) and \( v(x,t) = v(x-ct) \), we obtain the coupled ode system:

\[ Du'' + cu' - uv = 0 \]
\[ \lambda Dv'' + cv' + uv = 0 \] (8)

We now have a four dimensional system:

\[
\begin{align*}
\frac{du}{d\xi} & = z \\
D\frac{dz}{d\xi} & = -cz + uv \\
\frac{dv}{d\xi} & = w \\
\lambda D\frac{dw}{d\xi} & = -cw - uv
\end{align*}
\] (9)

We then get a Jacobian that looks like (in the order \( u, v, z, w \)):

\[
J = 
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
v/D & u/D & -c/D & 0 \\
-v/D\lambda & -u/D\lambda & 0 & -c/D\lambda
\end{pmatrix}
\] (10)

We observe that fixed points exist at \((0,0,0,0)\), \((m_1,0,0,0)\), and \((0,m_2,0,0)\), with \( m_1, m_2 \) arbitrary. First examine the Jacobian evaluated at \((0,0,0,0)\):

\[
J_{(0,0,0,0)} = 
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -c/D & 0 \\
0 & 0 & 0 & -c/D\lambda
\end{pmatrix}
\] (11)

The eigenvalues are 0 (double), \(-c/D\), and \(-c/D\lambda\). For the other two, we let \( m_1 = m_2 = K \), because that’s our boundary condition. This gives Jacobians of:

\[
J_{(K,0,0,0)} = 
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & K/D & -c/D & 0 \\
0 & -K/D\lambda & 0 & -c/D\lambda
\end{pmatrix}
\] (12)

and
Using MATLAB, Mathematica, or by hand, we find that the eigenvalues for the \((K,0,0,0)\) are \(0, -c/D\) and \(-c\pm\sqrt{c^2+4KD\lambda}\). For \((0,K,0,0)\) we similarly find \(0, -c/D\lambda\) and \(-c\pm\sqrt{c^2-4DK\lambda}\).

We have three special cases: where \(D \approx 0\), where \(\lambda \approx 0\), and where \(D\) is very small but \(\lambda\) is very large such that \(D\lambda \approx O(1)\). First examine when \(D \approx 0\). Then we have a 2D system:

\[
\begin{align*}
\frac{c}{\xi} \frac{du}{d\xi} &= uv \\
\frac{c}{\xi} \frac{dv}{d\xi} &= -uv
\end{align*}
\]  

with Jacobian

\[
\begin{pmatrix}
v/c & u/c \\
-v/c & -u/c
\end{pmatrix}
\]  

with eigenvalues 0 and \((u - v)/c\) and fixed points at \((K,0)\) and \((0,K)\).

For \(\lambda \approx 0\), we have a 3D system

\[
\begin{align*}
\frac{du}{d\xi} &= z \\
D \frac{dz}{d\xi} &= -cz + uv \\
\frac{c}{\xi} \frac{dv}{d\xi} &= -uv
\end{align*}
\]  

with eigenvalues 0 and \(\frac{1}{2(\lambda)}(uD + zc^2 \pm \sqrt{u^2D^2 - 2uDzc^2 + c^4z^2 + 4Dvc^2})\) with fixed points at \((0,K,0)\), \((K,0,0)\), and \((0,0,0)\).

Finally, we have the system where \(\lambda D \approx O(1)\) when \(D \approx 0\). This system is

\[
\begin{align*}
\frac{c}{\xi} \frac{du}{d\xi} &= uv \\
\frac{dv}{d\xi} &= w \\
\lambda D \frac{dw}{d\xi} &= -cw - uv
\end{align*}
\]  

with all the same fixed points as before.
3 We compute the "growth rate" $\eta$ for the Brusselator within a purely linearized treatment of the problem. First, compute $\eta$ at fixed $Q(\epsilon = 0) = a/\sqrt{D_a D_v}$. We have the coupled RDE's:

$$
\begin{align*}
    u_t &= D_u u_{xx} + f(u, v) = D_u u_{xx} + a - (1 + b)u + u^2 v \\
    v_t &= D_v v_{xx} + g(u, v) = D_v v_{xx} + bu - u^2 v
\end{align*}
$$

The fixed point occurs at $(u^*, v^*) = (a, b/a)$. Linearizing and Fourier-decomposing, we get

$$
J = \begin{pmatrix}
    f_u - q^2 D_u & f_v \\
    g_u - q^2 D_v & g_v
\end{pmatrix} = \begin{pmatrix}
    -(1 + b) - 2uv - q^2 D_u & u^2 \\
    b - 2uv & u^2 - q^2 D_v
\end{pmatrix} = \begin{pmatrix}
    b - 1 - q^2 D_u & a^2 \\
    -b & -a^2 - q^2 D_v
\end{pmatrix}
$$

Let $q = \pm Q$, so $Q^2 = \frac{D_u g_u + D_v f_u}{2D_u D_v} = -\frac{D_u a^2 + D_v (b-1)}{2D_u D_v}$. Additionally, note that the trace and determinant of the above Jacobian are $Tr = b - 1 - a^2 - q^2 (D_u + D_v)$ and $D = -bq^2 D_u + a^2 + q^2 D_v + q^2 a^2 D_u + q^4 D_u D_v$, respectively.

Now let's examine the growth rate at a fixed $Q$. Then $D_u Q^2 = c$, $D_v Q^2 = a/c$, and $(c + 1)^2 = b_T$. With $\epsilon = b - b_T$, we can simplify our trace and determinant to $Tr = (1 + \frac{1}{c})(c^2 - a^2) + \epsilon$ and $D = -\frac{\epsilon a^2}{c}$. Then

$$
\frac{T^2}{4} - D = \frac{(1 + 1/c)(c^2 - a^2)^2}{4} + \frac{\epsilon}{2} (c^2 + c - a^2 - \frac{a^2}{c}) + \epsilon a^2/c + \epsilon^2/4
$$

$$
= \frac{(1 + 1/c)(c^2 - a^2)^2}{2} + \frac{\epsilon}{2} (c^2(1 + 1/c) - a^2(1 - 1/c))
$$

$$
= \frac{(1 + 1/c)(c^2 - a^2)^2}{2} \left(1 + \frac{\epsilon}{2} \frac{4(c^2(1 + 1/c) - a^2(1 - 1/c))}{((1 + 1/c)(c^2 - a^2))^2}\right)
$$

so

$$
\sqrt{\frac{T^2}{4} - D} \approx \frac{(1 + 1/c)(c^2 - a^2)^2}{2} + \frac{\epsilon}{2} \frac{(c^2(1 + 1/c) - a^2(1 - 1/c))}{(1 + 1/c)(c^2 - a^2)}
$$

(21)

So the (-) eigenvalue gives us

$$
-\frac{T}{2} + \sqrt{\frac{T^2}{4} - D} = \frac{\epsilon a^2}{(c + 1)(a^2 - c^2)} = \frac{a^2(b - b_T)}{(c + 1)(a^2 - c^2)}
$$

(22)

which is what the notes have for (9.67).

Now we allow $Q$ to vary. Then

$$
Q^2(\epsilon) = \frac{D_u g_u + D_v f_u}{2D_u D_v} = -\frac{a^2 D_u + D_v (b - 1)}{2D_u D_v}
$$

(23)
So $D_a Q^2 = c + \epsilon/2$ and $D_v Q^2 = \frac{\alpha^2}{c}(1 + \epsilon/2c)$. Then

$$Tr = (1 + 1/c)(c^2 - a^2) + \frac{\epsilon}{2}(1 - a^2/c^2) \tag{24}$$

$$D = ((c+1)^2 + \epsilon - 1 - c - \frac{\epsilon}{2})(-a^2 - \frac{a^2}{c}(1 + \frac{\epsilon}{2c})) + a^2(\epsilon + (c+1)^2) = -\frac{a^2\epsilon}{c} - \frac{a^2\epsilon^2}{4c^2} \tag{25}$$

If this all gets plugged in to determine the eigenvalues, we get

$$\frac{-T}{2} - \sqrt{T/4 - D} \approx -\frac{1}{2}(1 + 1/c)(c^2 - a^2) - \frac{1}{2}(1 + 1/c)(a^2 - c^2)(1 + \frac{\epsilon}{2}(1 - a^2/c^2)(1 + 1/c)(c^2 - a^2) + 4a^2/c) - \frac{\epsilon}{4}(1 - a^2/c^2)$$

$$= \frac{a^2(b - tr)}{(c+1)(a^2 - c^2)} \tag{26}$$

4 We consider the real Ginsburg-Landau equation, $\psi_t = \mu \psi + \psi_{xx} - |\psi|^2 \psi$, with $\psi$ a complex field. We investigate the stability of static solutions of the form $\psi(x) = \sqrt{\mu - Q^2} e^{iQx}$. We first write $\psi(x,t) = \sqrt{\mu - Q^2} e^{iQx} + \eta(x)e^{i\lambda t} = \xi(x) + \eta(x)e^{i\lambda t}$. Then by only keeping terms that are linear in $\eta$ we get

$$\lambda \eta = \mu \eta + \eta_{xx} - 2|\xi|^2 \eta - \xi^2 \eta^* \tag{27}$$

We can define $\eta$ by its expansion

$$\eta = a_0 e^{iQx} + \sum (a_k e^{i(Q+k)x} + b_k e^{i(Q-k)x}) \tag{28}$$

Then

$$\lambda_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} \mu - (Q + k)^2 - 2(\mu - Q^2) \\ -(\mu - Q^2) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{29}$$

which gives us $\lambda_{k+} = -(\mu - Q^2) - k^2 \pm \sqrt{(2Qk)^2 + (\mu - Q^2)^2}$. To make $\xi(x)$ stable, we require the eigenvalue $\lambda_{k+}$ to be negative, which gives us our Eckhaus instability.