PHYSICS 221A : NONLINEAR DYNAMICS
HW ASSIGNMENT #3

(1) Riccati equations are nonlinear nonautonomous ODEs of the form

$$\frac{dx}{dt} = a(t)\,x^2 + b(t)\,x + c(t)\ .$$

(a) Make a change of dependent variable from $x(t)$ to $y(t)$, where

$$x(t) = -\frac{1}{a(t)}\,\frac{\dot{y}(t)}{y(t)}\ ,$$

which is known as the *Riccati transformation*. Show that $y(t)$ obeys a *linear* nonautonomous second order ODE. Write the formal solution to this ODE by writing it in the form $\dot{\boldsymbol{\varphi}} = M(t)\,\boldsymbol{\varphi}$ and expressing the solution in terms of a time ordered exponential.

(b) Solve the Riccati equation

$$\dot{x} = e^t\,x^2 - x + e^{-t}\ .$$

(c) Suppose we have a solution $X(t)$ to the Riccati equation. Show that by writing $x(t) = X(t) + u(t)$ we obtain the solvable Bernoulli equation

$$\dot{u} = a(t)\,u^2 + \Big(b(t) + 2\,a(t)X(t)\Big)\,u\ ,$$

which can then be solved using the method from problem (5) of homework set #1.

(d) Consider the Riccati equation

$$\dot{x} = x^2 - tx + 1\ .$$

By inspection, we have that $x(t) = t$ is a solution. Using the method of part (c) above, find a general solution for arbitrary $x(0) \equiv x_0$.

(2) Consider the equation

$$\ddot{x} + x = \epsilon\,x^5$$

with $\epsilon \ll 1$.

(a) Develop a two term straightforward expansion for the solution and discuss its uniformity.

(b) Using the Poincaré-Lindstedt method, find a uniformly valid expansion to first order.

(c) Using the multiple time scale method, find a uniformly valid expansion to first order.

(3) Consider the equation

$$\ddot{x} + \epsilon\,\dot{x}^3 + x = 0$$

with $\epsilon \ll 1$. Using the multiple time scale method, find a uniformly valid expansion to first order.

(4) Analyze the forced oscillator

$$\ddot{x} + x = \epsilon\left(\dot{x} - \tfrac{1}{3}\dot{x}^3\right) + \epsilon\, f_0\, \cos(t + \epsilon\nu t)$$

using the discussion in §4.3.1 and §4.3.2 of the notes as a template.

(5) Consider two coupled nonlinear oscillators, with

$$\frac{d\boldsymbol{\varphi}_1}{dt} = \boldsymbol{V}_1(\boldsymbol{\varphi}_1) + \epsilon\,\boldsymbol{F}_1(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \tag{1}$$

$$\frac{d\boldsymbol{\varphi}_2}{dt} = \boldsymbol{V}_2(\boldsymbol{\varphi}_2) + \epsilon\,\boldsymbol{F}_2(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\ . \tag{2}$$

Assume that for $\epsilon = 0$ each of the oscillators exhibits at least one stable limit cycle. Assume further that the natural frequencies $\omega_{1,2}$ of their respective limit cycles are close to resonance. That is, assume that the detuning

$$\nu = m\omega_1 - n\omega_2 \tag{3}$$

is small for some integer values of m and n.

(a) Using the phase representation and isochrones for each oscillator, show that to lowest order one may write

$$\frac{d\phi_1}{dt} = \omega_1 + \epsilon\, Q_1(\phi_1, \phi_2) \tag{4}$$

$$\frac{d\phi_2}{dt} = \omega_2 + \epsilon\, Q_2(\phi_1, \phi_2)\ . \tag{5}$$

(b) Expand the functions $Q_{1,2}(\phi_1, \phi_2)$ in a double Fourier series in their arguments. What terms satisfy the (near) resonance condition?

(c) Keeping only the terms which are nearly resonant, define $\psi = m\phi_1 - n\phi_2$ and derive an ODE describing the behavior of ψ. Analyze this ODE and classify its fixed points. What is the condition for synchronization of the two nonlinear oscillators?