Physics 4C — Final Solutions — Fall 2011

Problem 1

$$\vec{E} = \frac{E_0}{l} x\,\hat{x} + \frac{E_0}{l} y\,\hat{y} = E_x\hat{x} + E_y\hat{y}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\rho}{\varepsilon_0} \Rightarrow \frac{E_0}{l} + \frac{E_0}{l} = \frac{\rho}{\varepsilon_0} \Rightarrow$$

$$\boxed{\rho = \frac{2E_0\varepsilon_0}{l}} \qquad \rho = \frac{2}{2m}\cdot\frac{3N}{C}\cdot 8.85\times10^{-12}\frac{C^2}{Nm^2} \Rightarrow \boxed{\rho = 2.66\times10^{-11}\frac{C}{m^3}} \text{ (a)}$$

(b) Force on charge Q due to electric field is $F_E = Q\vec{E}$. We need to apply equal ~~and~~ opposite force to move the charge, so:

$\vec{F} = -Q\vec{E}$ does work $\quad W = \int_{\text{initial}}^{\text{final}} -Q\vec{E}\cdot d\vec{l}$

Move charge from $(0,0)$ to $(10l, 0)$ $\Rightarrow d\vec{l} = dx\,\hat{x}$

$$W = -Q\frac{E_0}{l}\int_0^{10l} dx\cdot x = -\frac{QE_0}{l}\cdot\frac{(10l)^2}{2} = -Q\cdot E_0\cdot 50l$$

So $\boxed{W = -Q\cdot E_0\cdot 50l}$ For $Q = -1.5C$, $E_0 = 3V/m$, $l = 2m$,

$W = +1.5\times3\times50\times2\ J \Rightarrow \boxed{W = 450\ J}$ (b)

$\boxed{\text{The work you do is } \underline{\text{positive}} \text{ (negative charge wants to stay at } (0,0))}$

(c) $$V(x,y) = -\int_{(0,0)}^{x,y}\vec{E}\cdot d\vec{l} = -\int\frac{E_0}{l}x'dx' - \frac{E_0}{l}y'dy' = -\frac{E_0}{2l}(x^2+y^2)$$

$$\boxed{V(x,y) = -\frac{E_0}{2l}(x^2+y^2)}$$

(d) Energy of dipole in field: $U = -\vec{p}\cdot\vec{E}$. Initially $U_i = -p\frac{E_0}{l}\cdot 5l$. In the $\hat{y}$ direction $U_f = 0$, so work done $= +p\cdot E_0\cdot 5 = 3C\cdot m\, 3\frac{N}{C}\cdot 5 = 45J$

$\boxed{\text{Work to rotate dipole} = 45\ J}$

Problem 2

$$\vec{E} = -\alpha y\,\hat{x} + \alpha x\,\hat{y} = E_x\,\hat{x} + E_y\,\hat{y}$$

(a) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$, $\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 + 0 = 0 \Rightarrow \boxed{\rho = 0}$

(b) $$\vec{\nabla}\times\vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\alpha y & \alpha x & 0 \end{pmatrix} = \hat{x}\cdot 0 + \hat{y}\cdot 0 + \hat{z}(\alpha+\alpha) = 2\alpha\,\hat{z}$$

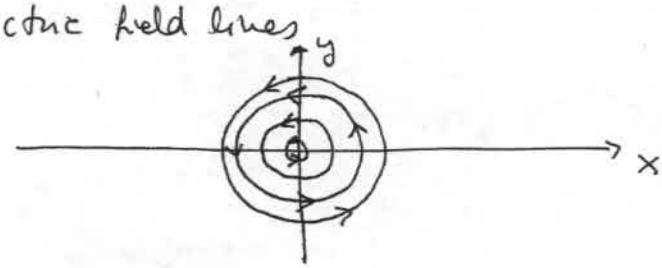
$$\boxed{\vec{\nabla}\times\vec{E} = 2\alpha\,\hat{z}}$$

(c) Use $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \frac{\partial\vec{B}}{\partial t} = -2\alpha\hat{z} \Rightarrow \boxed{\vec{B}(t) = -2\alpha t\,\hat{z}}$

When $t=2s$, $B = -2\alpha\cdot 2s = -2\cdot\frac{3\,V}{m^2}\cdot 2s = -12\,\frac{N}{C\cdot m}\cdot s = -12\,\frac{N}{A\cdot m} = -12\,T$

$\boxed{B = -12\,T \text{ for } t=2s}$ $\boxed{\text{points in the } -\hat{z} \text{ direction}}$

(d) Electric field lines



magnetic field points into the paper.

Problem 3

(a) (i) $r < a$, $E = 0$

(ii) $a < r < b$, $\vec{E} = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}$

(iii) $r > b$, $\vec{E} = \dfrac{2q}{4\pi\varepsilon_0 r^2}\hat{r}$

(b) $$U_1 = \int_a^b \frac{1}{2}\varepsilon_0 E^2 dV = \frac{1}{2}\cancel{\varepsilon_0}\cdot\cancel{4\pi}\cdot\frac{q^2}{(4\pi)^{\cancel{2}}\varepsilon_0^{\cancel{2}}}\int_a^b dr\,\frac{r^2}{r^4} =$$

$$= \frac{q^2}{8\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \boxed{\frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = U_1}$$

(c) $$U_2 = \int_b^\infty \frac{1}{2}\varepsilon_0 E^2 dV = \frac{1}{2}\varepsilon_0\cdot 4\pi\,\frac{(2q)^2}{(4\pi)^2\varepsilon_0^2}\int_b^\infty dr\,\frac{1}{r^2} =$$

$$= \frac{q^2}{2\pi\varepsilon_0}\int_b^\infty \frac{dr}{r^2} = \boxed{\frac{q^2}{2\pi\varepsilon_0}\,\frac{1}{b} = U_2}$$

(d) $$U_1 = U_2 \Rightarrow \frac{1}{8}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{2b} \Rightarrow \frac{4}{b} = \frac{1}{a} - \frac{1}{b} \Rightarrow$$

$$\Rightarrow \frac{5}{b} = \frac{1}{a} \Rightarrow b = 5a \Rightarrow \boxed{b/a = 5}$$

(e) Initial energy: $U_{in} = U_1 + U_2 = 2U_2 = \dfrac{q^2}{\pi\varepsilon_0 b}$

Final energy: $U_{fin} = \dfrac{q^2}{8\pi\varepsilon_0}\left(\dfrac{1}{b}+\dfrac{1}{a}\right) = \dfrac{q^2}{8\pi\varepsilon_0}\cdot\dfrac{6}{b} = \dfrac{3}{4}\dfrac{q^2}{\pi\varepsilon_0 b}$

Work done by you: $\boxed{W = U_{fin} - U_{in} = -\frac{1}{4}\frac{q^2}{\pi\varepsilon_0 b}}$

Work is negative because spheres repel so final energy is lower.

Problem 4

Clearly $\vec{E}$ points in z direction. $\vec{E} = E(z)\,\hat{z}$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E}{\partial z} = \frac{\rho}{\varepsilon_0} = \frac{\rho_0 z}{\varepsilon_0 l} \Rightarrow \boxed{E(z) = E(0) + \frac{\rho_0}{2\varepsilon_0 l} z^2} \quad (a)$$

(b) For ∞ sheet field outside is independent of distance

$\Rightarrow$ it is same on left side and right side $\Rightarrow$

$$E(z=0) = -E(z=l) \qquad \text{(they point in opposite directions)}$$

$$\Rightarrow -E(0) = E(0) + \frac{\rho_0}{2\varepsilon_0 l}\cdot l^2 \Rightarrow 2E(0) = \frac{\rho_0 l}{2\varepsilon_0} \Rightarrow$$

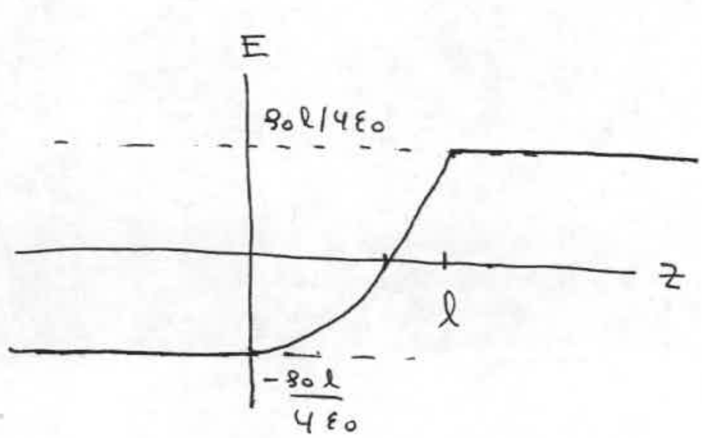
$$\boxed{E(0) = -\frac{\rho_0 l}{4\varepsilon_0}} \quad \boxed{E(l) = +\frac{\rho_0 l}{4\varepsilon_0}}$$

(c)
$$E(z) = -\frac{\cancel{\rho_0} l}{4\cancel{\varepsilon_0}} + \frac{\cancel{\rho_0}}{2\cancel{\varepsilon_0} l} z^2 = 0 \Rightarrow$$

$$\Rightarrow -\frac{l}{4} + \frac{z^2}{2l} = 0 \Rightarrow z^2 = \frac{l^2}{2} \Rightarrow z = \frac{l}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \boxed{z = 0.707\,l}$$

(d)



Problem 5

(a) Magnetic field from wire: $B = \frac{\mu_0 i}{2\pi r}$

$\Phi = \int \vec{B} \cdot d\vec{S}$; since $a \ll d$, B is approximately constant over area of loop.

$$\boxed{\Phi = B \cdot \pi a^2 \cdot \cos\theta = \frac{\mu_0 i \cancel{\pi} a^2 \cos\theta}{2\cancel{\pi} d}}$$

Φ is maximum when $\theta = 0$, loop is in plane of the paper, since magnetic field lines are perpendicular to the paper.

(b) Induced emf $\mathcal{E} = -\frac{d\Phi}{dt}$. Loop is rotating, so

$\boxed{\theta = \omega t} \Rightarrow \Phi(t) = \frac{\mu_0 i a^2}{2 d} \cos\omega t \Rightarrow$

$\Rightarrow \boxed{\mathcal{E}(t) = \frac{\mu_0 i a^2 \omega}{2 d} \sin\omega t}$ $\boxed{i_{ind}(t) = \frac{\mathcal{E}(t)}{R}}$

(c) Current is maximum for $\sin\theta = \pm 1 \Rightarrow$ loop is perpendicular to paper.
normal // to paper

If loop has self-inductance L, equation is

$$\mathcal{E}(t) - L\frac{di}{dt} - iR = 0.$$

(d) The inductance causes the current to lag the voltage.

So the maximum current is attained slightly later than when the self-inductance is ignored.

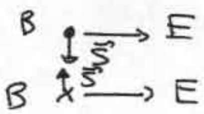
Problem 6

Ohm's law says $V = i \cdot R$, V is related to electric field through $V = E \cdot l$, and $\boxed{R = \frac{\rho \cdot l}{A} = \frac{\rho \cdot l}{\pi a^2}}$ $\Rightarrow$

$$E \cdot l = \frac{i \cdot \rho \cdot l}{\pi a^2} \Rightarrow \boxed{E = \frac{i \cdot \rho}{\pi a^2}} \quad (a)$$

(b) Magnetic field is $\boxed{B = \frac{\mu_0 i}{2\pi a}}$

(c) On lateral surfaces :



$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

E and B are perpendicular to each other, so magnitude of S is

$$S = \frac{1}{\mu_0} E B = \frac{1}{\cancel{\mu_0}} \frac{i \rho}{\pi a^2} \frac{\cancel{\mu_0} i}{2\pi a} = \frac{\rho i^2}{2\pi^2 a^3}$$

$$\boxed{S = \frac{\rho i^2}{2\pi^2 a^3}} \quad \boxed{\text{points radially inward}}$$

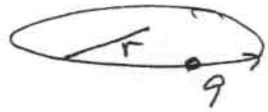
(d) $\int \vec{S} \cdot d\vec{A} = S \cdot A = S \cdot 2\pi a \cdot l = \frac{\rho i^2}{\cancel{2}\pi^{\cancel{2}} a^{\cancel{3}\,2}} \cdot \cancel{2\pi a} \cdot l = \frac{i^2 \rho l}{\pi a^2}$

so, $\boxed{\int \vec{S} \cdot d\vec{A} = i^2 \cdot \frac{\rho \cdot l}{\pi a^2} = i^2 \cdot R}$

(e) $\oint \vec{S} \cdot d\vec{A} = \int \vec{S} \cdot d\vec{A}$ since there is no flux through the top and bottom surfaces ($\vec{S}$ is parallel to these surfaces)

The law says that the change in energy per unit time is $-i^2 R$, that is precisely the energy per unit time (power) dissipated in the resistance, so it makes sense.

Problem 7



$$\mathcal{E} = -\frac{\partial \Phi}{\partial t}, \quad \Phi = \pi r^2 B(t) \Rightarrow$$

$$\boxed{\mathcal{E}(t) = -\pi r^2 \frac{dB}{dt}} \quad (a)$$

(b) $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 2\pi r E = -\pi r^2 \frac{dB}{dt} \Rightarrow \boxed{E = -\frac{r}{2}\frac{dB}{dt}}$

induced E-field is tangential to the orbit

(c) $\boxed{F = qE \text{ tangential to orbit}}$

(d) $F = m\frac{dv}{dt} \Rightarrow m\frac{dv}{dt} = qE = -\frac{qr}{2}\frac{dB}{dt}$

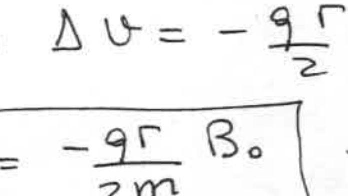
integrating both sides, $\int \frac{dB}{dt} = B(t) - B(0) = B_0 - 0 = B_0$

$$m\,\Delta v = -\frac{qr}{2} B_0 \quad \Rightarrow$$

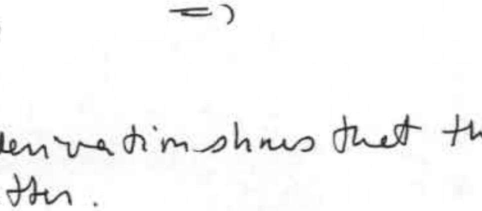
$$\boxed{\Delta v = -\frac{qr}{2m} B_0}$$

the derivation shows that the form of $B(t)$ doesn't matter.

(e) It depends on the direction of the initial velocity. If B is in the + z direction, assuming $q > 0$



q slows down



q speeds up

with $q < 0$ it is the other way around.

Problem 8

$$\vec{B} = B_0 [\cos(kz+\omega t) + \sin(kz+\omega t)]\,\hat{y}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B_y & 0 \end{pmatrix} = -\hat{x}\,\frac{\partial B_y}{\partial z} = -B_0 k\hat{x}\left[-\sin(kz+\omega t)+\cos(kz+\omega t)\right]$$

So $\vec{E} = E_x\,\hat{x}$; $\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} = B_0 k [\sin(kz+\omega t) - \cos(kz+\omega t)]$

$$\Rightarrow E_x = \frac{B_0 k}{\mu_0 \varepsilon_0}\,\frac{1}{\omega}\left[-\sin(kz+\omega t) - \cos(kz+\omega t)\right]$$

use $\frac{1}{\mu_0\varepsilon_0} = c^2$, $\frac{k}{\omega} = \frac{1}{c}$ $\Rightarrow$

$$\boxed{\vec{E} = -cB_0[\sin(kz+\omega t) + \cos(kz+\omega t)]\,\hat{x}}$$

(b) Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

$$\boxed{\text{Direction: } -\hat{x}\times\hat{y} = -\hat{z}}$$

Magnitude:

$$\boxed{S = \frac{cB_0^2}{\mu_0}\left[\sin(kz+\omega t) + \cos(kz+\omega t)\right]^2}$$