PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #6 SOLUTIONS

(1) $\nu = 8$ moles of a diatomic ideal gas are subjected to a cyclic quasistatic process, the thermodynamic path for which is an ellipse in the (V, p) plane. The center of the ellipse lies at $(V_0, p_0) = (0.25\,\text{m}^3, 1.0\,\text{bar})$. The semimajor axes of the ellipse are $\Delta V = 0.10\,\text{m}^3$ and $\Delta p = 0.20\,\text{bar}$.

(a) What is the temperature at $(V, p) = (V_0 + \Delta V, p_0)$?

(b) Compute the net work per cycle done by the gas.

(c) Compute the internal energy difference $E(V_0 - \Delta V, p_0) - E(V_0, p_0 - \Delta p)$.

(d) Compute the heat Q absorbed by the gas along the upper half of the cycle.

Solution :

(a) The temperature is $T = pV/\nu R$. With $V = V_0 + \Delta V = 0.35\,\text{m}^3$ and $p = p_0 = 1.0\,\text{bar}$, we have

$$T = \frac{(10^5\,\text{Pa})(0.35\,\text{m}^3)}{(8\,\text{mol})(8.31\,\text{J/mol K})} = 530\,\text{K} .$$

(b) The area of an ellipse is π times the product of the semimajor axis lengths.

$$\oint p\,dV = \pi\,(\Delta p)(\Delta V) = \pi\,(0.20 \times 10^6\,\text{bar})\,(0.10\,\text{m}^3) = 6.3\,\text{kJ} .$$

(c) For a diatomic ideal gas, $E = \frac{5}{2}pV$. Thus,

$$\Delta E = \tfrac{5}{2}\big(V_0\,\Delta p - p_0\,\Delta V\big) = \tfrac{5}{2}\,(-0.05 \times 10^5\,\text{J}) = -13\,\text{kJ} .$$

(d) We have $Q = \Delta E + W$, with

$$W = 2\,p_0\,\Delta V + \tfrac{\pi}{2}(\Delta p)(\Delta V) = 23\,\text{kJ} ,$$

which is the total area under the top half of the ellipse. The difference in energy is given by $\Delta E = \frac{5}{2}\,p_0 \cdot 2\Delta V = 5\,p_0\,\Delta V$, so

$$Q = \Delta E + W = 7\,p_0\,\Delta V + \tfrac{\pi}{2}(\Delta p)(\Delta V) = 73\,\text{kJ} .$$

(2) Determine which of the following differentials are exact and which are inexact.

(a) $xy\,dx + xy\,dy$

(b) $(x + y^{-1})\,dx - xy^{-2}\,dy$

(c) $xy^3\,dx + 3x^2y^2\,dy$

(d) $(\ln y + \ln z)\,dx + xy^{-1}\,dy + xz^{-1}dz$

Solution :

Recall $đF = \sum_i A_i(\boldsymbol{x})\,dx_i$ is exact if $\frac{\partial A_i}{\partial x_j} = \frac{\partial A_j}{\partial x_i}$ for all i and j. Thus,

(a) $đF = xy\,dx + xy\,dy$ is inexact, since $\partial A_x/\partial y = x$ but $\partial A_y/\partial x = y$. However, $đF = xy\,d(x + y)$, $(xy)^{-1}đF = d(x + y)$ is exact.

(b) $đF = (x + y^{-1})\,dx - xy^{-2}\,dy = d\big(\frac{1}{2}x^2 + xy^{-1}\big)$ is exact.

(c) $đF = xy^3\,dx + 3x^2y^2\,dy$ is inexact, since $\partial A_x/\partial y = 3xy^2$ but $\partial A_y/\partial x = 6xy^2$. However, $đF = x\,d(xy^3)$, so $x^{-1}đF = d(xy^3)$ is exact.

(d) $đF = (\ln y + \ln z)\,dx + xy^{-1}\,dy + xz^{-1}dz = d(x\ln y + x\ln z)$ is exact.

(3) Liquid mercury at atmospheric pressure and temperature $T = 0^\circ$ C has a molar volume of $14.72\,\text{cm}^3/\text{mol}$ and a specific heat a constant pressure of $c_p = 28.0\,\text{J/mol·K}$. Its coefficient of expansion is $\alpha = \frac{1}{V}\big(\frac{\partial V}{\partial T}\big)_p = 1.81 \times 10^{-4}/\text{K}$ and its isothermal compressibility is $\kappa_T = -\frac{1}{V}\big(\frac{\partial V}{\partial T}\big)_T = 3.88 \times 10^{-12}\,\text{cm}^2/\text{dyn}$. Find its specific heat at constant volume c_V and the ratio $\gamma = c_p/c_V$. [Reif problem 5.10]

Solution :

According to eqn. (2.307) in the notes,

$$c_p - c_V = \frac{vT\alpha_p^2}{\kappa_T} = \frac{(14.72 \times 10^{-6}\text{m}^3/\text{mol})(273\,\text{K})(1.81 \times 10^{-4}/\text{K})^2}{3.88 \times 10^{-11}\,\text{m}\,\text{s}^2/\text{kg}} = 3.39\,\text{J/K}\ .$$

Thus,

$$c_V = 24.6\,\text{J/K} \qquad , \qquad \gamma = \frac{c_p}{c_v} = \frac{28.0}{24.6} = 1.14\ .$$

(4) ν moles of an ideal diatomic gas are driven along the cycle depicted in Fig. 1. Section AB is an adiabatic free expansion; section BC is an isotherm at temperature $T_A = T_B = T_C$; CD is an isobar, and DA is an isochore. The volume at B is given by $V_B = (1 - x) V_A + x V_C$, where $0 \le x \le 1$.

(a) Find an expression for the total work W_{cycle} in terms of ν, T_A, V_A, V_C, and x.

(b) Suppose $V_A = 1.0\,\text{L}$, $V_C = 5.0\,\text{L}$, $T_A = 500\,\text{K}$, and $\nu = 5$. What is the volume V_B such that $W_{\text{cycle}} = 0$?

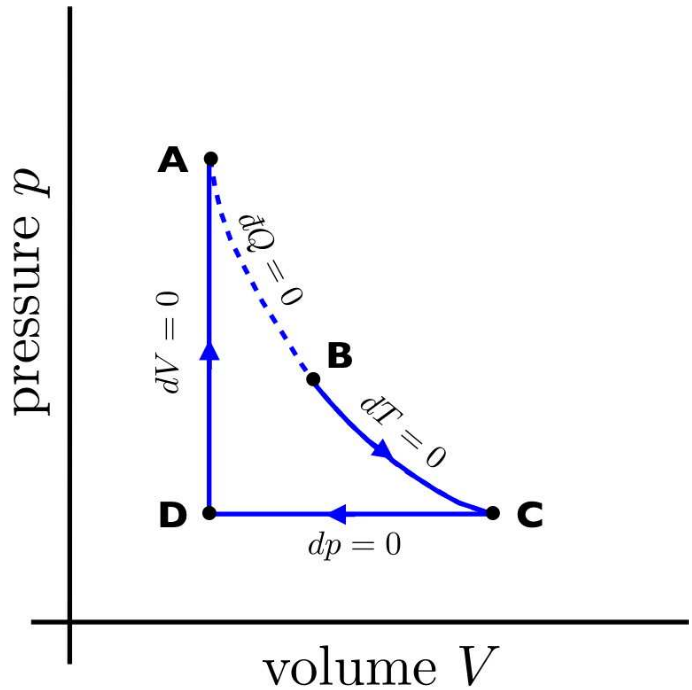

Figure 1: Thermodynamic cycle for problem 4, consisting of adiabatic free expansion (AB), isotherm (BC), isobar (CD), and isochore (DA).

Solution :

(a) We have $W_{AB} = W_{DA} = 0$, and

$$W_{BC} = \int_B^C p\, dV = \nu R T_A \int_B^C \frac{dV}{V} = \nu R T_A \ln\left(\frac{V_C}{V_B}\right)$$

$$W_{CD} = \int_C^D p\, dV = p_C (V_D - V_C) = -\nu R T_A \left(1 - \frac{V_A}{V_C}\right) .$$

Thus,

$$W_{CYC} = \nu R T_A \left[\ln\left(\frac{V_C}{V_B}\right) - 1 + \frac{V_A}{V_C} \right] .$$

(b) Setting $V_{\mathsf{B}} = (1-x)\,V_{\mathsf{A}} + x\,V_{\mathsf{C}}$, and defining $r \equiv V_{\mathsf{A}}/V_{\mathsf{C}}$, we have

$$W_{\mathsf{CYC}} = \nu R\, T_{\mathsf{A}} \Big(-\ln\big(x + (1-x)\,r\big) + 1 - r \Big) ,$$

and setting $W_{\mathsf{CYC}} = 0$ we obtain $x = x^*$, with

$$x^* = \frac{e^{r-1} - r}{1 - r} .$$

For $V_{\mathsf{A}} = 1.0\,\mathrm{L}$ and $V_{\mathsf{C}} = 5.0\,\mathrm{L}$, we have $r = \frac{1}{5}$ and $x^* = 0.31$, corresponding to $V_{\mathsf{B}} = 2.2\,\mathrm{L}$.

(5) A strange material found stuck to the bottom of a seat in Warren Lecture Hall 2001 obeys the thermodynamic relation $E(S, V, N) = a\,S^6/V^2N^3$, where a is a dimensionful constant.

(a) What are the MKS dimensions of a?

(b) Find the equation of state relating p, V, N, and T.

(c) Find the coefficient of thermal expansion $\alpha = \frac{1}{V}\big(\frac{\partial V}{\partial T}\big)_p$. Express your answer in terms of intensive quantities p, T, and $n = N/V$.

(d) Find the isothermal compressibility $\kappa = -\frac{1}{V}\big(\frac{\partial V}{\partial p}\big)_T$. Express your answer in terms of intensive quantities p, T, and $n = N/V$.

Solution :

(a) From $[E] = \mathrm{J}$, $[S] = \mathrm{J/K}$, and $[V] = \mathrm{m}^3$, we obtain $[a] = \mathrm{m}^6\,\mathrm{K}^5/\mathrm{J}^5$.

(b) We have

$$T = \left(\frac{\partial E}{\partial S}\right)_{VN} = \frac{6aS^5}{V^2N^3} \qquad , \qquad p = -\left(\frac{\partial E}{\partial V}\right)_{SN} = \frac{2aS^6}{V^3N^3} .$$

We can eliminate S by finding the ratio T^6/p^5:

$$\frac{T^6}{p^5} = \frac{6^6}{2^5} \cdot \frac{aV^3}{N^3} = 1458\,a\,n^{-3} .$$

This is an equation of state, which we can recast as

$$p(T, n) = \frac{T^{6/5}\,n^{3/5}}{(1458\,a)^{1/5}} .$$

Contrast this with the ideal gas law, $p = nk_{\mathrm{B}}T$. For parts (c) and (d) it is useful to take the logarithm, and obtain

$$6\,\ln T = 5\,\ln p + 3\,\ln V - 3\,\ln N + \ln(1458\,a) .$$

(c) The coefficient of volume expansion is

$$\alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{pN} = \left(\frac{\partial \ln V}{\partial T}\right)_{pN} = \frac{2}{T}\,.$$

(d) The isothermal compressibility is

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{TN} = -\left(\frac{\partial \ln V}{\partial p}\right)_{TN} = \frac{5}{3\,p}\,.$$