PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #2 SOLUTIONS

(1) Consider the matrix

$$M = \begin{pmatrix} 4 & 4 \\ -1 & 9 \end{pmatrix} .$$

(a) Find the characteristic polynomial $P(\lambda) = \det(\lambda \mathbb{I} - M)$ and the eigenvalues.

(b) For each eigenvalue λ_α, find the associated right eigenvector R_i^α and left eigenvector L_i^α. Normalize your eigenvectors so that $\langle\, L^\alpha \,|\, R^\beta \,\rangle = \delta_{\alpha\beta}$.

(c) Show explicitly that $M_{ij} = \sum_\alpha \lambda_\alpha \, R_i^\alpha \, L_j^\alpha$.

Solution :

(a) The characteristic polynomial is

$$P(\lambda) = \det\begin{pmatrix} \lambda - 4 & -4 \\ 1 & \lambda - 9 \end{pmatrix} = \lambda^2 - 13\,\lambda + 40 = (\lambda - 5)(\lambda - 8) ,$$

so the two eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 8$.

(b) Let us write the right eigenvectors as $\vec{R}^\alpha = \begin{pmatrix} R_1^\alpha \\ R_2^\alpha \end{pmatrix}$ and the left eigenvectors as $\vec{L}^\alpha = \begin{pmatrix} L_1^\alpha & L_2^\alpha \end{pmatrix}$. Having found the eigenvalues, we only need to solve four equations:

$$4R_1^1 + 4R_2^1 = 5R_1^1 \quad , \quad 4R_1^2 + 4R_2^2 = 8R_1^2 \quad , \quad 4L_1^1 - L_2^1 = 5L_1^1 \quad , \quad 4L_1^2 - L_2^2 = 8L_1^2 \, .$$

We are free to choose $R_1^\alpha = 1$ when possible. We must also satisfy the normalizations $\langle\, L^\alpha \,|\, R^\beta \,\rangle = L_i^\alpha R_i^\beta = \delta^{\alpha\beta}$. We then find

$$\vec{R}^1 = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} \quad , \quad \vec{R}^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad \vec{L}^1 = \begin{pmatrix} \frac{4}{3} & -\frac{4}{3} \end{pmatrix} \quad , \quad \vec{L}^2 = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \end{pmatrix} \, .$$

(c) The projectors onto the two eigendirections are

$$P_1 = |\, R^1 \,\rangle\langle\, L^1 \,| = \begin{pmatrix} \frac{4}{3} & -\frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \quad , \quad P_2 = |\, R^2 \,\rangle\langle\, L^2 \,| = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{pmatrix} \, .$$

Note that $P_1 + P_2 = \mathbb{I}$. Now construct

$$\lambda_1 \, P_1 + \lambda_2 \, P_2 = \begin{pmatrix} 4 & 4 \\ -1 & 9 \end{pmatrix} ,$$

as expected.

(2) A *Markov chain* is a probabilistic process which describes the transitions of discrete stochastic variables in time. Let $P_i(t)$ be the probability that the system is in state i at time t. The time evolution equation for the probabilities is

$$P_i(t+1) = \sum_j Y_{ij}\, P_j(t) \ .$$

Thus, we can think of $Y_{ij} = P(i\,,\,t+1\,|\,j\,,\,t)$ as the *conditional probability* that the system is in state i at time $t+1$ given hat it was in state j at time t. Y is called the *transition matrix*. It must satisfy $\sum_i Y_{ij} = 1$ so that the total probability $\sum_i P_i(t)$ is conserved.

Suppose I have two bags of coins. Initially bag A contains two quarters and bag B contains five dimes. Now I do an experiment. Every minute I exchange a random coin chosen from each of the bags. Thus the number of coins in each bag does not fluctuate, but their values do fluctuate.

(a) Label all possible states of this system, consistent with the initial conditions. (*I.e.* there are always two quarters and five dimes shared among the two bags.)

(b) Construct the transition matrix Y_{ij}.

(c) Show that the total probability is conserved is $\sum_i Y_{ij} = 1$, and verify this is the case for your transition matrix Y. This establishes that $(1, 1, \ldots, 1)$ is a left eigenvector of Y corresponding to eigenvalue $\lambda = 1$.

(d) Find the eigenvalues of Y.

(e) Show that as $t \to \infty$, the probability $P_i(t)$ converges to an equilibrium distribution $P_i^{\rm eq}$ which is given by the right eigenvector of i corresponding to eigenvalue $\lambda = 1$. Find $P_i^{\rm eq}$, and find the long time averages for the value of the coins in each of the bags.

Solution :

(a) There are three possible states consistent with the initial conditions. In state $|\,1\,\rangle$, bag A contains two quarters and bag B contains five dimes. In state $|\,2\,\rangle$, bag A contains a quarter and a dime while bag B contains a quarter and five dimes. In state $|\,3\,\rangle$, bag A contains two dimes while bag B contains three dimes and two quarters. We list these states in the table below, along with their degeneracies. The degeneracy of a state is the number of configurations consistent with the state label. Thus, in state $|\,2\,\rangle$ the first coin in bag A could be a quarter and the second a dime, or the first could be a dime and the second a quarter. For bag B, any of the five coins could be the quarter.

(b) To construct Y_{ij}, note that transitions out of state $|\,1\,\rangle$, *i.e.* the elements Y_{i1}, are particularly simple. With probability 1, state $|\,1\,\rangle$ always evolves to state $|\,2\,\rangle$. Thus, $Y_{21} = 1$ and $Y_{11} = Y_{31} = 0$. Now consider transitions out of state $|\,2\,\rangle$. To get to state $|\,1\,\rangle$, we need to choose the D from bag A (probability $\frac{1}{2}$) and the Q from bag B (probability $\frac{1}{5}$). Thus,

$Y_{12} = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$. For transitions back to state $|\,2\,\rangle$, we could choose the Q from bag A (probability $\frac{1}{2}$) if we also chose the Q from bag B (probability $\frac{1}{5}$). Or we could choose the D from bag A (probability $\frac{1}{2}$) and one of the D's from bag B (probability $\frac{4}{5}$). Thus, $Y_{22} = \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{1}{2}$. Reasoning thusly, one obtains the transition matrix,

$$Y = \begin{pmatrix} 0 & \frac{1}{10} & 0 \\ 1 & \frac{1}{2} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix} .$$

Note that $\sum_i Y_{ij} = 1$.

$\lvert\, j\,\rangle$	bag A	bag B	$g_j^{\rm A}$	$g_j^{\rm B}$	$g_j^{\rm TOT}$
$\lvert\,1\,\rangle$	QQ	DDDDD	1	1	1
$\lvert\,2\,\rangle$	QD	DDDDQ	2	5	10
$\lvert\,3\,\rangle$	DD	DDDQQ	1	10	10

Table 1: States and their degeneracies.

(c) Our explicit form for Y confirms the sum rule $\sum_i Y_{ij} = 1$ for all j. Thus, $\vec{L}^1 = (1\;1\;1)$ is a left eigenvector of Y with eigenvalue $\lambda = 1$.

(d) To find the other eigenvalues, we compute the characteristic polynomial of Y and find, easily,

$$P(\lambda) = \det(\lambda\,\mathbb{I} - Y) = \lambda^3 - \tfrac{11}{10}\,\lambda^2 + \tfrac{1}{25}\,\lambda + \tfrac{3}{50} \,.$$

This is a cubic, however we already know a root, *i.e.* $\lambda = 1$, and we can explicitly verify $P(\lambda = 1) = 0$. Thus, we can divide $P(\lambda)$ by the monomial $\lambda - 1$ to get a quadratic function, which we can factor. One finds after a small bit of work,

$$\frac{P(\lambda)}{\lambda - 1} = \lambda^2 - \tfrac{3}{10}\,\lambda - \tfrac{3}{50} = \big(\lambda - \tfrac{3}{10}\big)\big(\lambda + \tfrac{1}{5}\big) \,.$$

Thus, the eigenspectrum of Y is $\lambda_1 = 1$, $\lambda_2 = \frac{3}{10}$, and $\lambda_3 = -\frac{1}{5}$.

(e) We can decompose Y into its eigenvalues and eigenvectors, like we did in problem (1). Write

$$Y_{ij} = \sum_{\alpha=1}^{3} \lambda_\alpha R_i^\alpha\, L_j^\alpha \,.$$

Now let us start with initial conditions $P_i(0)$ for the three configurations. We can always decompose this vector in the right eigenbasis for Y, *viz.*

$$P_i(t) = \sum_{\alpha=1}^{3} C_\alpha(t)\, R_i^\alpha \;,$$

The initial conditions are $C_\alpha(0) = \sum_i L_i^\alpha \, P_i(0)$. But now using our eigendecomposition of Y, we find that the equations for the discrete time evolution for each of the C_α decouple:

$$C_\alpha(t+1) = \lambda_\alpha C_\alpha(t) \; .$$

Clearly as $t \to \infty$, the contributions from $\alpha = 2$ and $\alpha = 3$ get smaller and smaller, since $C_\alpha(t) = \lambda_\alpha^t \, C_\alpha(0)$, and both λ_2 and λ_3 are smaller than unity in magnitude. Thus, as $t \to \infty$ we have $C_1(t) \to C_1(0)$, and $C_{2,3}(t) \to 0$. Note $C_1(0) = \sum_i L_i^1 \, P_i(0) = \sum_i P_i(0) = 1$, since $\vec{L}^1 = (1\;1\;1)$. Thus, we obtain $P_i(t \to \infty) \to R_i^1$, the components of the eigenvector $\vec{R}^1$. It is not too hard to explicitly compute the eigenvectors:

$$\vec{L}^1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad \vec{L}^2 = \begin{pmatrix} 10 & 3 & -4 \end{pmatrix} \qquad \vec{L}^3 = \begin{pmatrix} 10 & -2 & 1 \end{pmatrix}$$

$$\vec{R}^1 = \tfrac{1}{21} \begin{pmatrix} 1 \\ 10 \\ 10 \end{pmatrix} \qquad \vec{R}^2 = \tfrac{1}{35} \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \qquad \vec{R}^3 = \tfrac{1}{15} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \; .$$

Thus, the equilibrium distribution $P_i^{\rm eq} = \lim_{t \to \infty} P_i(t)$ satisfies detailed balance:

$$P_j^{\rm eq} = \frac{g_j^{\rm TOT}}{\sum_l g_l^{\rm TOT}} \; .$$

Working out the average coin value in bags A and B under equilibrium conditions, one finds $A = \frac{200}{7}$ and $B = \frac{500}{7}$ (centa), and B/A is simply the ratio of the number of coins in bag B to the number in bag A. Note $A + B = 100$ cents, as the total coin value is conserved.

(3) Poincar'e recurrence is guaranteed for phase space dynamics that are *invertible*, *volume preserving*, and acting on a *bounded phase space*.

(a) Give an example of a map which is volume preserving on a bounded phase space, but which is not invertible and not recurrent.

(b) Give an example of a map which is invertible on a bounded phase space, but which is not volume preserving and not recurrent.

(c) Give an example of a map which is invertible and volume preserving, but on an unbounded phase space and not recurrent.

Solution :

(a) Consider the map $f(x) = \text{frac}(x)$, where $\text{frac}(x) = x - \text{gint(x)}$ is the fractional part of x, obtained by subtracting from x the greatest integer less than x. Acting on any set of width less than unity, this map is volume-preserving. However it is many-to-one hence not invertible. For example, $f(\pi) = f(\pi - 1) = f(\pi - 2) = \pi - 3$. For sufficiently small ϵ, the interval $[\pi - \epsilon \, , \, \pi + \epsilon]$ gets mapped onto the interval $[\pi - 3 - \epsilon \, , \, \pi - 3 + \epsilon]$, never to return to the original interval.

(b) Any dissipative dynamical system will do. For example, consider $\dot{x} = p/m$, $\dot{p} = -\gamma p$, on some finite region of (x, p) space which contains the origin.

(c) Consider $\dot{x} = p/m$, $\dot{p} = 0$ on the infinite phase space $(x, p) \in \mathbb{R}^2$. If $p \neq 0$ the x-motion is monotonically increasing or decreasing (*i.e.* either to the right or to the left along the real line).

(4) Consider a toroidal phase space $(x, p) \in \mathbb{T}^2$. You can describe the torus as a square $[0, 1] \times [0, 1]$ with opposite sides identified. Design your own modified Arnold cat map acting on this phase space, *i.e.* a 2×2 matrix with integer coefficients and determinant 1.

(a) Start with an initial distribution localized around the center – say a disc centered at $(\frac{1}{2}, \frac{1}{2})$. Show how these initial conditions evolve under your map. Can you tell whether your dynamics are mixing?

(b) Now take a pixelated image. For reasons discussed in the lecture notes, this image should exhibit Poincaré recurrence. Can you see this happening?

Solution :

(a) Any map

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{M} \begin{pmatrix} x \\ p \end{pmatrix} ,$$

will due, provided $\det M = ad - bc = 1$. Arnold's cat map has $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Consider the generalized cat map with $M = \begin{pmatrix} 1 & 1 \\ p & p+1 \end{pmatrix}$. Starting from an initial square distribution, we iterate the map up to three times and show the results in Figs. 1, 3, and 5. The numerical results are consistent with a mixing flow. (With just a few further interations, almost the entire torus is covered.)

(c) A pixelated image exhibits Poincaré recurrence, as we see in Figs. 2, 4, and 6.

(5) Consider a spin singlet formed by two $S = \frac{1}{2}$ particles, $|\,\Psi\,\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow_\mathsf{A}\downarrow_\mathsf{B}\rangle - |\downarrow_\mathsf{A}\uparrow_\mathsf{B}\rangle\big)$. Find the reduced density matrix, $\rho_\mathsf{A} = \mathsf{Tr}_\mathsf{B}\,|\,\Psi\,\rangle\langle\,\Psi\,|$.

Solution :

We have

$$|\,\Psi\,\rangle\langle\,\Psi\,| = \tfrac{1}{2}\,|\uparrow_\mathsf{A}\downarrow_\mathsf{B}\rangle\langle\uparrow_\mathsf{A}\downarrow_\mathsf{B}| + \tfrac{1}{2}\,|\downarrow_\mathsf{A}\uparrow_\mathsf{B}\rangle\langle\downarrow_\mathsf{A}\uparrow_\mathsf{B}| - \tfrac{1}{2}\,|\uparrow_\mathsf{A}\downarrow_\mathsf{B}\rangle\langle\downarrow_\mathsf{A}\uparrow_\mathsf{B}| - \tfrac{1}{2}\,|\downarrow_\mathsf{A}\uparrow_\mathsf{B}\rangle\langle\uparrow_\mathsf{A}\downarrow_\mathsf{B}| \,.$$

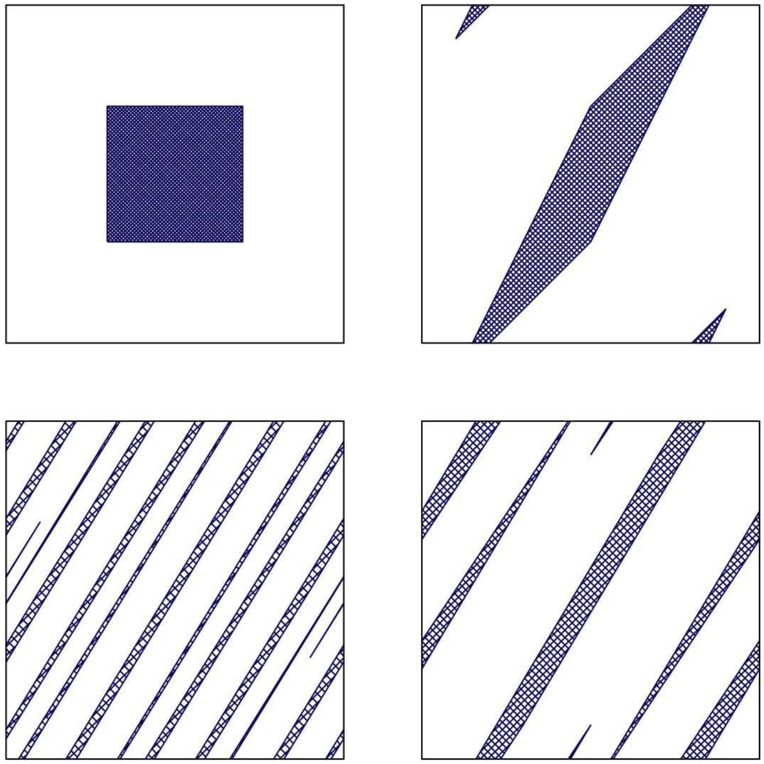

Figure 1: Zeroth, first, second, and third iterates of the generalized cat map with $p = 1$ (*i.e.* Arnold's cat map), acting on an initial square distribution (clockwise from upper left).

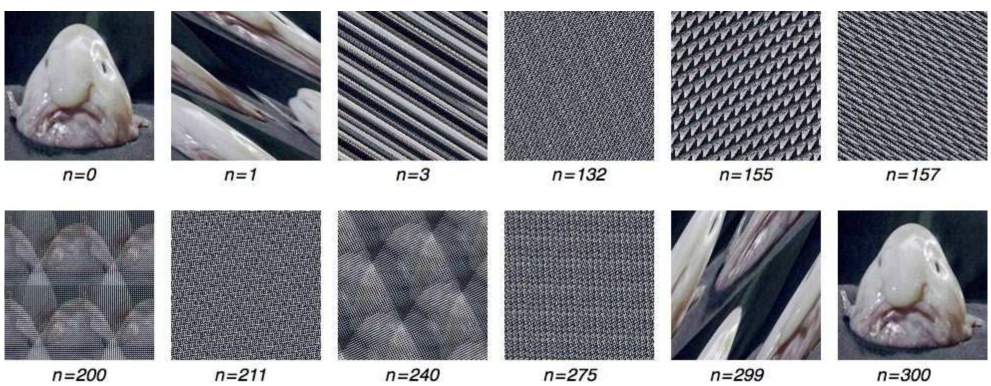

Figure 2: Evolution of a pixelated blobfish under the Arnold cat map.

Now take the trace over the spin degrees of freedom on site B. Only the first two terms contribute, resulting in the reduced density matrix

$$\rho_{\mathsf{A}} = \operatorname*{Tr}_{\mathsf{B}} |\,\Psi\,\rangle\langle\,\Psi\,| = \tfrac{1}{2}\,|\uparrow_{\mathsf{A}}\rangle\langle\uparrow_{\mathsf{A}}| + \tfrac{1}{2}\,|\downarrow_{\mathsf{A}}\rangle\langle\downarrow_{\mathsf{A}}|\;.$$

Note that $\mathsf{Tr}\,\rho_{\mathsf{A}} = 1$, but whereas the full density matrix $\rho = \mathsf{Tr}_{\mathsf{B}}\,|\,\Psi\,\rangle\langle\,\Psi\,|$ had one eigenvalue of 1, corresponding to eigenvector $|\,\Psi\,\rangle$, and three eigenvalues of 0 (any state or-

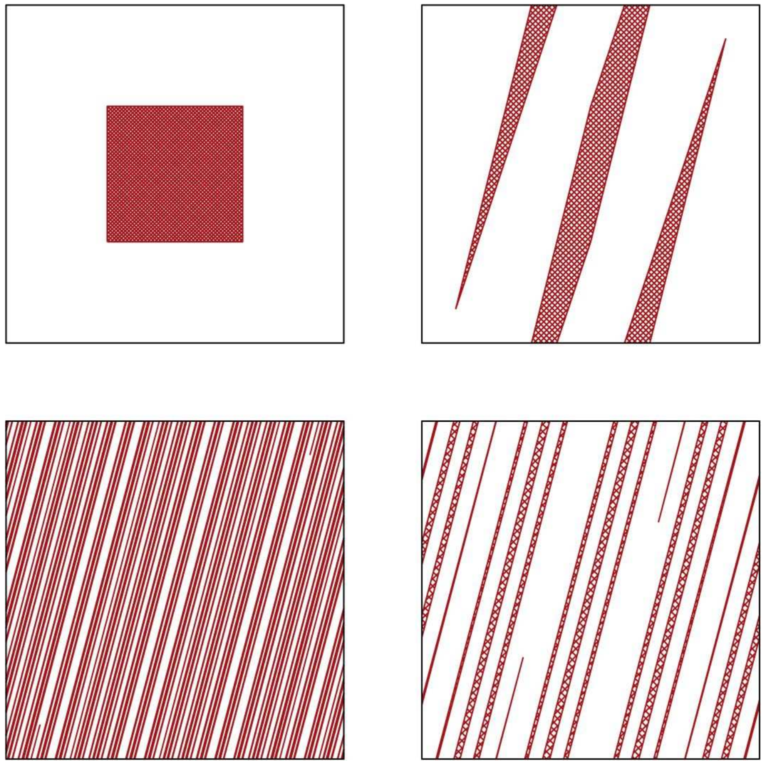

Figure 3: Zeroth, first, second, and third iterates of the generalized cat map with $p = 2$, acting on an initial square distribution (clockwise from upper left).

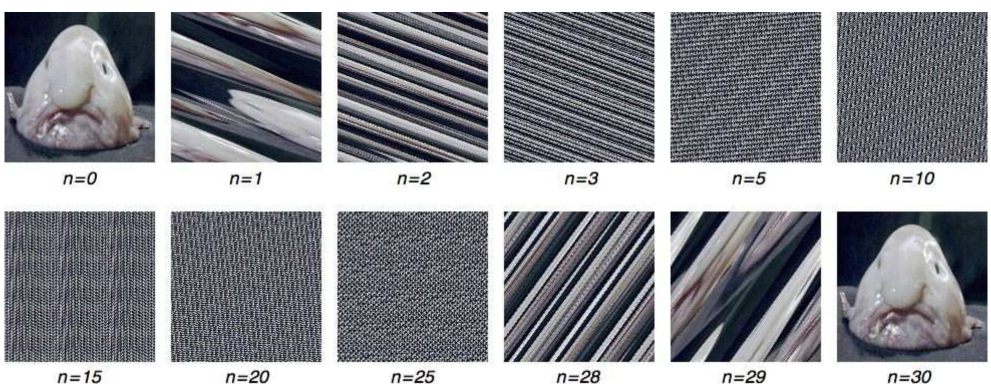

Figure 4: Evolution of a pixelated blobfish under the $p = 2$ generalized cat map.

thogonal to $|\,\Psi\,\rangle$, the reduced density matrix ρ_{A} does not correspond to a 'pure state' in that it is not a projector. It has two degenerate eigenvalues at $\lambda = \frac{1}{2}$. The quantity $S_{\mathsf{A}} = -\,\mathsf{Tr}\,\rho_{\mathsf{A}} \ln \rho_{\mathsf{A}} = \ln 2$ is the *quantum entanglement entropy* for the spin singlet.

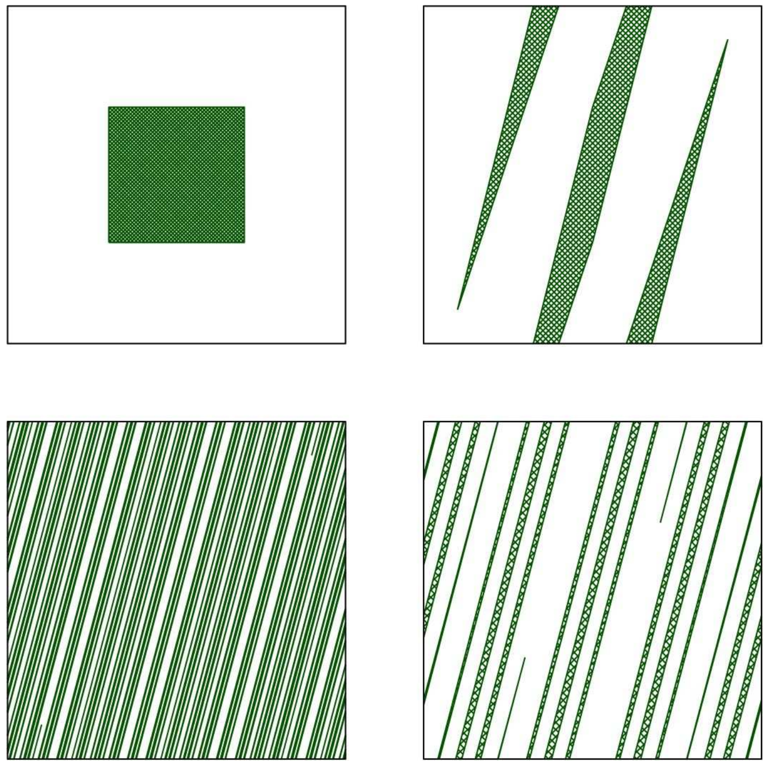

Figure 5: Zeroth, first, second, and third iterates of the generalized cat map with $p = 3$, acting on an initial square distribution (clockwise from upper left).

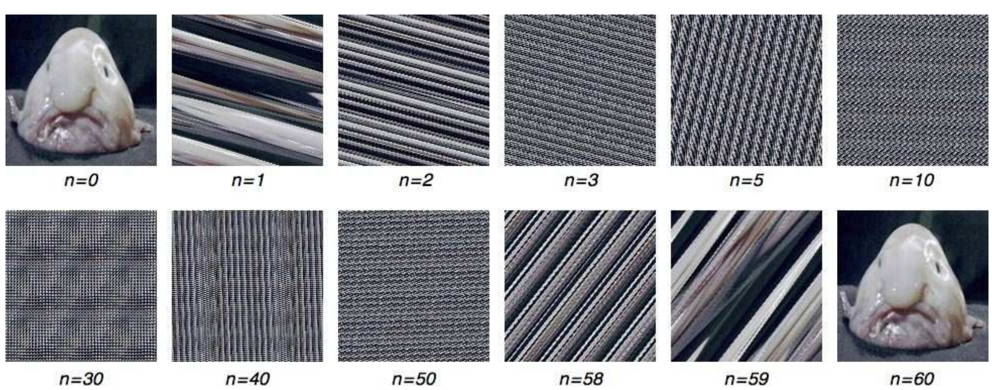

Figure 6: Evolution of a pixelated blobfish under the $p = 3$ generalized cat map.