PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #1 SOLUTIONS

(**1**) Consider the contraption in Fig. 1. At each of k steps, a particle can fork to either the left ($n_j = 1$) or to the right ($n_j = 0$). The final location is then a k-digit binary number.

(a) Assume the probability for moving to the left is p and the probability for moving to the right is $q \equiv 1 - p$ at each fork, independent of what happens at any of the other forks. *I.e.* all the forks are uncorrelated. Compute $\langle X_k \rangle$. *Hint: X_k can be represented as a k-digit binary number, i.e.* $X_k = n_{k-1}n_{k-2}\cdots n_1 n_0 = \sum_{j=0}^{k-1} 2^j n_j$.

(b) Compute $\langle X_k^2 \rangle$ and the variance $\langle X_k^2 \rangle - \langle X_k \rangle^2$.

(c) X_k may be written as the sum of k random numbers. Does X_k satisfy the central limit theorem as $k \to \infty$? Why or why not?

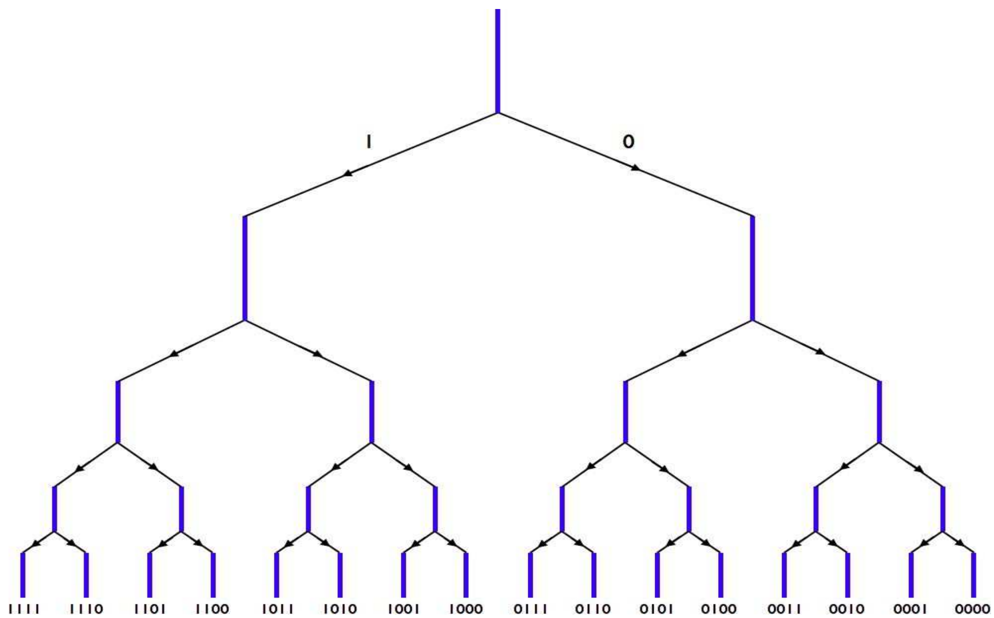

Figure 1: Generator for a k-digit random binary number ($k = 4$ shown).

Solution :

(a) The position after k forks can be written as a k-digit binary number: $n_{k-1}n_{k-2}\cdots n_1 n_0$. Thus,

$$X_k = \sum_{j=0}^{k-1} 2^j \, n_j \; ,$$

where $n_j = 0$ or 1 according to $P_n = p\,\delta_{n,1} + q\,\delta_{n,0}$. Now it is clear that $\langle n_j \rangle = p$, and therefore

$$\langle X_k \rangle = p \sum_{j=0}^{k-1} 2^j = p \cdot \left(2^k - 1\right) .$$

(b) The variance in X_k is

$$\begin{aligned}\mathrm{Var}(X_k) = \langle X_k^2 \rangle - \langle X_k \rangle^2 &= \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} 2^{j+j'} \Big(\langle n_j n_{j'} \rangle - \langle n_j \rangle \langle n_{j'} \rangle \Big) \\ &= p(1-p) \sum_{j=0}^{k-1} 4^j = p(1-p) \cdot \tfrac{1}{3} \left(4^k - 1\right) ,\end{aligned}$$

since $\langle n_j n_{j'} \rangle - \langle n_j \rangle \langle n_{j'} \rangle = p(1-p)\, \delta_{jj'}$.

(c) Clearly the distribution of X_k does not obey the CLT, since $\langle X_k \rangle$ scales exponentially with k. Also note

$$\lim_{k\to\infty} \frac{\sqrt{\mathrm{Var}(X_k)}}{\langle X_k \rangle} = \sqrt{\frac{1-p}{3p}} ,$$

which is a constant. For distributions obeying the CLT, the ratio of the rms fluctuations to the mean scales as the inverse square root of the number of trials. The reason that this distribution does not obey the CLT is that the variance of the individual terms is increasing with j.

(**2**) Let $P(x) = (2\pi\sigma^2)^{-1/2}\, e^{-(x-\mu)^2/2\sigma^2}$. Compute the following integrals:

(a) $I = \int\limits_{-\infty}^{\infty} dx\; P(x)\, x^3$.

(b) $I = \int\limits_{-\infty}^{\infty} dx\; P(x)\, \cos(Qx)$.

(c) $I = \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy\; P(x)\, P(y)\, e^{xy}$. You may set $\mu = 0$ to make this somewhat simpler. Under what conditions does this expression converge?

Solution :

(a) Write

$$x^3 = (x - \mu + \mu)^3 = (x-\mu)^3 + 3(x-\mu)^2 \mu + 3(x-\mu)\mu^2 + \mu^3 ,$$

so that

$$\langle x^3 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{-\infty}^{\infty} dt\; e^{-t^2/2\sigma^2} \Big\{ t^3 + 3t^2\mu + 3t\mu^2 + \mu^3 \Big\} .$$

Since $\exp(-t^2/2\sigma^2)$ is an even function of t, odd powers of t integrate to zero. We have $\langle t^2\rangle = \sigma^2$, so

$$\langle x^3\rangle = \mu^3 + 3\mu\sigma^2 \ .$$

A nice trick for evaluating $\langle t^{2k}\rangle$:

$$\langle t^{2k}\rangle = \frac{\int\limits_{-\infty}^{\infty} dt\, e^{-\lambda t^2}\, t^{2k}}{\int\limits_{-\infty}^{\infty} dt\, e^{-\lambda t^2}} = \frac{(-1)^k \frac{d^k}{d\lambda^k} \int\limits_{-\infty}^{\infty} dt\, e^{-\lambda t^2}}{\int\limits_{-\infty}^{\infty} dt\, e^{-\lambda t^2}} = \frac{(-1)^k}{\sqrt{\lambda}}\, \frac{d^k\sqrt{\lambda}}{d\lambda^k}\bigg|_{\lambda=1/2\sigma^2}$$

$$= \tfrac{1}{2}\cdot\tfrac{3}{2}\cdots\tfrac{(2k-1)}{2}\,\lambda^{-k}\big|_{\lambda=1/2\sigma^2} = \frac{(2k)!}{2^k\,k!}\,\sigma^{2k} \ .$$

(b) We have

$$\langle \cos(Qx)\rangle = \mathsf{Re}\,\langle e^{iQx}\rangle = \mathsf{Re}\left[\frac{e^{iQ\mu}}{\sqrt{2\pi\sigma^2}}\int\limits_{-\infty}^{\infty} dt\, e^{-t^2/2\sigma^2}\, e^{iQt}\right]$$

$$= \mathsf{Re}\left[e^{iQ\mu}\, e^{-Q^2\sigma^2/2}\right] = \cos(Q\mu)\, e^{-Q^2\sigma^2/2} \ .$$

Here we have used the result

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int\limits_{-\infty}^{\infty} dt\, e^{-\alpha t^2 - \beta t} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

with $\alpha = 1/2\sigma^2$ and $\beta = -iQ$. Another way to do it is to use the general result derive above in part (a) for $\langle t^{2k}\rangle$ and do the sum:

$$\langle \cos(Qx)\rangle = \mathsf{Re}\,\langle e^{iQx}\rangle = \mathsf{Re}\left[\frac{e^{iQ\mu}}{\sqrt{2\pi\sigma^2}}\int\limits_{-\infty}^{\infty} dt\, e^{-t^2/2\sigma^2}\, e^{iQt}\right]$$

$$= \cos(Q\mu)\sum_{k=0}^{\infty}\frac{(-Q^2)^k}{(2k)!}\,\langle t^{2k}\rangle = \cos(Q\mu)\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\tfrac{1}{2}Q^2\sigma^2\right)^k$$

$$= \cos(Q\mu)\, e^{-Q^2\sigma^2/2} \ .$$

(c) We have

$$I = \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy\, P(x)\, P(y)\, e^{\kappa^2 xy} = \frac{e^{-\mu^2/2\sigma^2}}{2\pi\sigma^2}\int d^2\!x\, e^{-\frac{1}{2}A_{ij}\, x_i\, x_j}\, e^{b_i\, x_i} \ ,$$

where $\boldsymbol{x} = (x, y)$,

$$A = \begin{pmatrix} \sigma^2 & -\kappa^2 \\ -\kappa^2 & \sigma^2 \end{pmatrix} \quad , \quad \boldsymbol{b} = \begin{pmatrix} \mu/\sigma^2 \\ \mu/\sigma^2 \end{pmatrix} \ .$$

Using the general formula for the Gaussian integral,

$$\int d^n x\, e^{-\frac{1}{2}A_{ij}\,x_i\,x_j}\, e^{b_i\,x_i} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}}\,\exp\left(\tfrac{1}{2}A^{-1}_{ij}\,b_i\,b_j\right)\,,$$

we obtain

$$I = \frac{1}{\sqrt{1-\kappa^4\sigma^4}}\,\exp\left(\frac{\mu^2\kappa^2}{1-\kappa^2\sigma^2}\right)\,.$$

Convergence requires $\kappa^2\sigma^2 < 1$.

(**3**) The binomial distribution,

$$B_N(n,p) = \binom{N}{n}\,p^n\,(1-p)^{N-n}\,,$$

tells us the probability for n successes in N trials if the individual trial success probability is p. The average number of successes is $\nu = \sum_{n=0}^N n\,B_N(n,p) = Np$. Consider the limit $N \to \infty$.

(a) Show that the probability of n successes becomes a function of n and ν alone. That is, evaluate

$$P_\nu(n) = \lim_{N\to\infty} B_N(n,\nu/N)\,.$$

This is the *Poisson distribution*.

(b) Show that the moments of the Poisson distribution are given by

$$\langle n^k\rangle = e^{-\nu}\left(\nu\,\frac{\partial}{\partial\nu}\right)^k e^\nu\,.$$

(c) Evaluate the mean and variance of the Poisson distribution.

The Poisson distribution is also known as the *law of rare events* since $p = \nu/N \to 0$ in the $N \to \infty$ limit. See `http://en.wikipedia.org/wiki/Poisson_distribution#Occurrence` for some amusing applications of the Poisson distribution.

Solution :

(a) We have

$$P_\nu(n) = \lim_{N\to\infty}\frac{N!}{n!\,(N-n)!}\left(\frac{\nu}{N}\right)^n\left(1-\frac{\nu}{N}\right)^{N-n}\,.$$

Note that

$$(N-n)! \simeq (N-n)^{N-n}\,e^{n-N} = N^{N-n}\left(1-\frac{n}{N}\right)^N e^{n-N} \to N^{N-n}\,e^N\,,$$

where we have used the result $\lim_{N\to\infty}\big(1+\frac{x}{N}\big)^N = e^x$. Thus, we find

$$P_\nu(n) = \frac{1}{n!}\,\nu^n\,e^{-\nu}\ ,$$

the Poisson distribution. Note that $\sum_{n=0}^{\infty} P_n(\nu) = 1$ for any ν.

(b) We have

$$\begin{aligned}\langle n^k\rangle &= \sum_{n=0}^{\infty} P_\nu(n)\,n^k = \sum_{n=0}^{\infty}\frac{1}{n!}\,n^k\nu^n\,e^{-\nu}\\ &= e^{-\nu}\Big(\nu\,\frac{d}{d\nu}\Big)^k\sum_{n=0}^{\infty}\frac{\nu^n}{n!} = e^{-\nu}\,\Big(\nu\,\frac{\partial}{\partial\nu}\Big)^k\,e^{\nu}\ .\end{aligned}$$

(c) Using the result from (b), we have $\langle n\rangle = \nu$ and $\langle n^2\rangle = \nu+\nu^2$, hence $\mathrm{Var}(n)=\nu$.

(4) Consider a D-dimensional *random walk* on a hypercubic lattice. The position of a particle after N steps is given by

$$\boldsymbol{R}_N = \sum_{j=1}^{N}\hat{\boldsymbol{n}}_j\ ,$$

where $\hat{\boldsymbol{n}}_j$ can take on one of $2D$ possible values: $\hat{\boldsymbol{n}}_j\in\big\{\pm\,\hat{\boldsymbol{e}}_1,\ldots,\pm\hat{\boldsymbol{e}}_D\big\}$, where $\hat{\boldsymbol{e}}_\mu$ is the unit vector along the positive x_μ axis. Each of these possible values occurs with probability $1/2D$, and each step is statistically independent from all other steps.

(a) Consider the *generating function* $S_N(\boldsymbol{k}) = \big\langle e^{i\boldsymbol{k}\cdot\boldsymbol{R}_N}\big\rangle$. Show that

$$\big\langle R_N^{\alpha_1}\cdots R_N^{\alpha_J}\big\rangle = \frac{1}{i}\frac{\partial}{\partial k_{\alpha_1}}\cdots\frac{1}{i}\frac{\partial}{\partial k_{\alpha_J}}\bigg|_{\boldsymbol{k}=0} S_N(\boldsymbol{k})\ .$$

For example, $\langle R_N^\alpha R_N^\beta\rangle = -\big(\partial^2 S_N(\boldsymbol{k})/\partial k_\alpha\partial k_\beta\big)_{\boldsymbol{k}=0}$.

(b) Evaluate $S_N(\boldsymbol{k})$ for the case $D=3$ and compute the quantities $\langle X_N^4\rangle$ and $\langle X_N^2\,Y_N^2\rangle$.

Solution :

(a) The result follows immediately from

$$\begin{aligned}\frac{1}{i}\frac{\partial}{\partial k_\alpha}\,e^{i\boldsymbol{k}\cdot\boldsymbol{R}} &= R_\alpha\,e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\\ \frac{1}{i}\frac{\partial}{\partial k_\alpha}\,\frac{1}{i}\frac{\partial}{\partial k_\beta}\,e^{i\boldsymbol{k}\cdot\boldsymbol{R}} &= R_\alpha\,R_\beta\,e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\ ,\end{aligned}$$

et cetera. Keep differentiating with respect to the various components of $\boldsymbol{k}$.

(b) For $D = 3$, there are six possibilities for $\hat{\boldsymbol{n}}_j$: $\pm\hat{\boldsymbol{x}}$, $\pm\hat{\boldsymbol{y}}$, and $\pm\hat{\boldsymbol{z}}$. Each occurs with a probability $\frac{1}{6}$, independent of all the other $\hat{\boldsymbol{n}}_{j'}$ with $j' \neq j$. Thus,

$$
\begin{aligned}
S_N(\boldsymbol{k}) &= \prod_{j=1}^{N} \langle e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{n}}_j} \rangle = \left[\frac{1}{6}\Big(e^{ik_x} + e^{-ik_x} + e^{ik_y} + e^{-ik_y} + e^{ik_z} + e^{-ik_z}\Big)\right]^N \\
&= \left(\frac{\cos k_x + \cos k_y + \cos k_z}{3}\right)^N .
\end{aligned}
$$

We have

$$
\begin{aligned}
\langle X_N^4 \rangle = \frac{\partial^4 S(\boldsymbol{k})}{\partial k_x^4}\bigg|_{\boldsymbol{k}=0} &= \frac{\partial^4}{\partial k_x^4}\bigg|_{k_x=0} \big(1 - \tfrac{1}{6}\,k_x^2 + \tfrac{1}{72}\,k_x^4 + \ldots\big)^N \\
&= \frac{\partial^4}{\partial k_x^4}\bigg|_{k_x=0} \Big[1 + N\big(-\tfrac{1}{6}\,k_x^2 + \tfrac{1}{72}\,k_x^4 + \ldots\big) + \tfrac{1}{2}N(N-1)\big(-\tfrac{1}{6}\,k_x^2 + \tfrac{1}{72}\,k_x^4 + \ldots\big)^2 + \ldots\Big] \\
&= \frac{\partial^4}{\partial k_x^4}\bigg|_{k_x=0} \Big[1 - \tfrac{1}{6}Nk_x^2 + \tfrac{1}{72}N^2k_x^4 + \ldots\Big] = \tfrac{1}{3}N^2 .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\langle X_N^2\, Y_N^2 \rangle = \frac{\partial^4 S(\boldsymbol{k})}{\partial k_x^2\,\partial k_y^2}\bigg|_{\boldsymbol{k}=0} &= \frac{\partial^4}{\partial k_x^2\,\partial k_y^2}\bigg|_{k_x=0} \big(1 - \tfrac{1}{6}\,(k_x^2 + k_y^2) + \tfrac{1}{72}\,(k_x^4 + k_y^4) + \ldots\big)^N \\
&= \frac{\partial^4}{\partial k_x^2\,\partial k_y^2}\bigg|_{k_x=k_y=0} \Big[1 + N\big(-\tfrac{1}{6}\,(k_x^2 + k_y^2) + \tfrac{1}{72}\,(k_x^4 + k_y^4) + \ldots\big) + \tfrac{1}{2}N(N-1)\big(-\tfrac{1}{6}\,(k_x^2 + k_y^2) + \ldots\big)^2 + \ldots\Big] \\
&= \frac{\partial^4}{\partial k_x^2\,\partial k_y^2}\bigg|_{k_x=k_y=0} \Big[1 - \tfrac{1}{6}N(k_x^2 + k_y^2) + \tfrac{1}{72}N^2(k_x^4 + k + y^4) + \tfrac{1}{36}\,k_x^2\,k_y^2 + \ldots\Big] = \tfrac{1}{9}N(N-1) .
\end{aligned}
$$

(**5**) A rare disease is known to occur in $f = 0.02\%$ of the general population. Doctors have designed a test for the disease with $\nu = 99.90\%$ sensitivity and $\rho = 99.95\%$ specificity.

(a) What is the probability that someone who tests positive for the disease is actually sick?

(b) Suppose the test is administered twice, and the results of the two tests are independent. If a random individual tests positive both times, what are the chances he or she actually has the disease?

(c) For a binary partition of events, find an expression for $P(X|A \cap B)$ in terms of $P(A|X)$, $P(B|X)$, $P(A|\neg X)$, $P(B|\neg X)$, and the priors $P(X)$ and $P(\neg X) = 1 - P(X)$. You should assume A and B are independent, so $P(A \cap B|X) = P(A|X) \cdot P(B|X)$.

Solution :

(a) Let X indicate that a person is infected, and A indicate that a person has tested positive. We then have $\nu = P(A|X) = 0.9990$ is the sensitivity and $\rho = P(\neg A|\neg X) = 0.9995$ is the specificity. From Bayes' theorem, we have

$$P(X|A) = \frac{P(A|X) \cdot P(X)}{P(A|X) \cdot P(X) + P(A|\neg X) \cdot P(\neg X)} = \frac{\nu f}{\nu f + (1-\rho)(1-f)} \;,$$

where $P(A|\neg X) = 1 - P(\neg A|\neg X) = 1 - \rho$ and $P(X) = f$ is the fraction of infected individuals in the general population. With $f = 0.0002$, we find $P(X|A) = 0.2856$.

(b) We now need

$$P(X|A^2) = \frac{P(A^2|X) \cdot P(X)}{P(A^2|X) \cdot P(X) + P(A^2|\neg X) \cdot P(\neg X)} = \frac{\nu^2 f}{\nu^2 f + (1-\rho)^2(1-f)} \;,$$

where A^2 indicates two successive, independent tests. We find $P(X|A^2) = 0.9987$.

(c) Assuming A and B are independent, we have

$$\begin{aligned} P(X|A \cap B) &= \frac{P(A \cap B|X) \cdot P(X)}{P(A \cap B|X) \cdot P(X) + P(A \cap B|\neg X) \cdot P(\neg X)} \\ &= \frac{P(A|X) \cdot P(B|X) \cdot P(X)}{P(A|X) \cdot P(B|X) \cdot P(X) \; + P(A|\neg X) \cdot P(B|\neg X) \cdot P(\neg X)} \,. \end{aligned}$$

This is exactly the formula used in part (b).

(**6**) Compute the entropy of the F08 Physics 140A grade distribution (in bits). The distribution is available from http://physics.ucsd.edu/students/courses/fall2008/physics140. Assume 11 possible grades: A+, A, A-, B+, B, B-, C+, C, C-, D, F.

Solution :

Assuming the only possible grades are A+, A, A-, B+, B, B-, C+, C, C-, D, F (11 possibilities), then from the chart we produce the entries in Tab. 1. We then find

$$S = -\sum_{n=1}^{11} p_n \log_2 p_n = 2.82 \text{ bits}$$

The maximum information content would be for $p_n = \frac{1}{11}$ for all n, leading to $S_{\max} = \log_2 11 = 3.46$ bits.

$\sum_n N_n = 38$	A+	A	A-	B+	B	B-	C+	C	C-	D	F
N_n	2	9	7	3	9	3	1	2	0	2	0
$-p_n \log_2 p_n$	0.224	0.492	0.450	0.289	0.492	0.289	0.138	0.224	0	0.224	0

Table 1: F08 Physics 140A final grade distribution.