Physics 130B Final Examination Solutions Fall 2011

1. (20 pt) Consider a Simple Harmonic Oscillator in standard notation: $\hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2$. Add a small perturbation, bx^3, to the potential energy.

(a 5) Find the 1st-order energy shifts for all states, |*n*>.

$E_n^{(1)} = \langle n|b\hat{x}^3|n\rangle$, *and* $\hat{x}^3 \propto \left(a^\dagger + a\right)^3$, which has 3 raising/lowering operators in each term. An odd number of raising/lowering operators can never couple to the starting state |n>, so all 1st-order E-shifts are 0.

Another way: all the |*n*> are even about $x = 0$ in $\psi^*\psi$, and x^3 is odd. Therefore the integral is always 0:

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,x^3\,dx = 0\,.$$

(b 10) Find the 1st-order state changes for all states, $|\psi_n^{(1)}\rangle$.

$$\left|\psi_n^{(1)}\right\rangle = \sum_{k\neq n} \frac{\langle k|b\hat{x}^3|n\rangle}{\varepsilon_n - \varepsilon_k}|k\rangle. \qquad \textit{Since} \quad \hat{x}^3 \propto \left(a^\dagger + a\right)^3, k = n \pm 1 \text{ or } \pm 3. \textit{ This makes } n = 0,1,2 \textit{ special}$$

$$\left(a^\dagger + a\right)^3 = \left(a^\dagger\right)^3 + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + aaa \ (8 \textit{ terms})$$

We could leave it like this, but it's easier to combine a's and $a^\dagger$'s into n-hat operators:

$$\textit{Use}: a^\dagger a = \hat{n},\ a a^\dagger = \hat{n}+1: \ \left(a^\dagger + a\right)^3 = \left(a^\dagger\right)^3 + a^\dagger\hat{n} + a^\dagger\left(\hat{n}+1\right) + \hat{n}a + \left(\hat{n}+1\right)a^\dagger + a\hat{n} + a\left(\hat{n}+1\right) + aaa$$

$$\textit{Now convert the } \hat{n} \textit{ to numbers, not operators.} \ \ NB: \hat{n}a = (n-1)a,\ \hat{n}a^\dagger = (n+1)a^\dagger$$

$$\left(a^\dagger + a\right)^3 = \left(a^\dagger\right)^3 + (3n+3)a^\dagger + (3n)a + a^3$$

$$\left|\psi_n^{(1)}\right\rangle = b\left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\times\left[\underbrace{\frac{\sqrt{n(n-1)(n-2)}}{3\hbar\omega}|n-3\rangle}_{\textit{from } a^3} + \underbrace{\frac{3n\sqrt{n}}{\hbar\omega}|n-1\rangle}_{\textit{from } 3na} + \underbrace{\frac{(3n+3)\sqrt{n+1}}{-\hbar\omega}|n+1\rangle}_{\textit{from } (3n+3)a^\dagger} + \underbrace{\frac{\sqrt{(n+3)(n+2)(n+1)}}{-3\hbar\omega}|n+3\rangle}_{\textit{from } \left(a^\dagger\right)^3}\right]$$

For $n = 0$, only the last 2 terms exist. For $n = 1, 2$, only the last 3 terms exist.

(c 5) Find the average position <x> for the ground state, $|\psi_0\rangle \approx |0\rangle + \left|\psi_0^{(1)}\right\rangle$.

$$|\psi_0\rangle \approx |0\rangle + \left|\psi_0^{(1)}\right\rangle = |0\rangle + b\left(\frac{\hbar}{2m\omega}\right)^{3/2}\frac{3}{-\hbar\omega}|1\rangle + b\left(\frac{\hbar}{2m\omega}\right)^{3/2}\frac{\sqrt{6}}{-3\hbar\omega}|3\rangle,$$

$$\langle\psi_0|\hat{x}|\psi_0\rangle = \langle\psi_0|\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)|\psi_0\rangle = b\left(\frac{\hbar}{2m\omega}\right)^2\left(\frac{3}{-\hbar\omega}\langle 1|a^\dagger|0\rangle + \frac{3}{-\hbar\omega}\langle 0|a|1\rangle\right) + \textit{zero terms}$$

$$= -\frac{6b}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^2$$

2. (20 pt) A system has 2 spinless particles in different $l = 1$ orbitals (thus no exclusion principle applies).

(a 5) What are all possible quantum numbers for the total orbital angular momentum, L?

Note: L is the *magnitude* quantum number, and always $L \ge 0$. $L = |l_1 - l_2|, ...0, ...l_1 + l_2 = 0, 1, 2$

(b 5) Write the state |2 2> in the uncoupled basis, i.e., in terms of $|l_1\ m_1;\ l_2\ m_2\rangle$ basis states.

The only way to get $M = 2$ is for $m_1 = m_2 = 1$: $\left|2\,2\right\rangle = \left|1\,1;1\,1\right\rangle$

(c 5) Write the state |2 1> in the uncoupled basis.

Lower the above equation, using the lowering operator in the coupled basis on the left, and uncoupled basis on the right (dropping ħ's, since they cancel):

$$L_-\left|2\,2\right\rangle = \left(l_{1-} + l_{2-}\right)\left|1\,1;1\,1\right\rangle \qquad Use: \quad L_-\left|L\,M\right\rangle = \hbar\sqrt{L(L+1) - M(M-1)}$$

$$\sqrt{2\cdot 3 - 2\cdot 1}\left|2\,1\right\rangle = \sqrt{1\cdot 2 - 1\cdot 0}\left|1\,0;1\,1\right\rangle + \sqrt{1\cdot 2 - 1\cdot 0}\left|1\,1;1\,0\right\rangle$$

$$\sqrt{4}\left|2\,1\right\rangle = \sqrt{2}\left|1\,0;1\,1\right\rangle + \sqrt{2}\left|1\,1;1\,0\right\rangle \quad \Rightarrow \quad \left|2\,1\right\rangle = \frac{1}{\sqrt{2}}\left|1\,0;1\,1\right\rangle + \frac{1}{\sqrt{2}}\left|1\,1;1\,0\right\rangle$$

Note that this is, and must be, normalized. We have no freedom to introduce normalization factors.

Or, from the symmetry of the two particles, they must have the same CG coefficient, and the result must be normalized, so we conclude without calculation that the CGs are both 1/√2.

(d 5) Write the state |2 0> in the uncoupled basis.

Lower the result of part (c):

$$L_-\left|2\,1\right\rangle = \frac{1}{\sqrt{2}}\left(l_{1-} + l_{2-}\right)\left[\left|1\,0;1\,1\right\rangle + \left|1\,1;1\,0\right\rangle\right] \qquad (\text{dropping the } \hbar\text{'s, since they cancel})$$

$$\sqrt{6-0}\left|2\,0\right\rangle = \frac{1}{\sqrt{2}}\left[\left(\sqrt{2-0}\left|1\,-1;1\,1\right\rangle + \sqrt{2-0}\left|1\,0;1\,0\right\rangle\right) + \left(\sqrt{2-0}\left|1\,0;1\,0\right\rangle + \sqrt{2-0}\left|1\,1;1\,-1\right\rangle\right)\right]$$

$$\sqrt{6}\left|2\,0\right\rangle = \left|1\,-1;1\,1\right\rangle + 2\left|1\,0;1\,0\right\rangle + \left|1\,1;1\,-1\right\rangle \quad \Rightarrow \quad \left|2\,0\right\rangle = \frac{1}{\sqrt{6}}\left|1\,-1;1\,1\right\rangle + \sqrt{\frac{2}{3}}\left|1\,0;1\,0\right\rangle + \frac{1}{\sqrt{6}}\left|1\,1;1\,-1\right\rangle$$

Again, the result is automatically normalized.

3. (15 pt) A silver atom has its spin along the axis $\theta = 15°$, $\phi = 45°$.

(a 3) What is the probability of measuring spin up along the z-axis?

$$\left|\chi\right\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{+i\phi} \end{pmatrix} = \begin{pmatrix} \cos 7.5 \\ \sin 7.5 e^{+i\pi/4} \end{pmatrix} \qquad \Pr\left(\left|z+\right\rangle\right) = |a|^2 = \cos^2 7.5$$

(b 3) What is the probability of measuring spin down along the z-axis?

$$\Pr\left(\left|z-\right\rangle\right) = |b|^2 = \sin^2 7.5 \quad Or \quad \Pr\left(\left|z-\right\rangle\right) = 1 - \Pr\left(\left|z+\right\rangle\right) = 1 - \cos^2 7.5 = \sin^2 7.5$$

(c 3) What is the average spin component along the z-axis?

$$\left\langle s_z \right\rangle = +\frac{\hbar}{2}\cos^2 7.5 - \frac{\hbar}{2}\sin^2 7.5 = \frac{\hbar}{2}\left(\cos^2 7.5 - \sin^2 7.5\right) = \frac{\hbar}{2}\cos 15$$

This must equal the classical component of s_z in part (d) below.

(d 3) What is the classical component of spin along the z-axis, for this spin orientation?

This is just the projection of the tilted-z-component, $\hbar/2$, onto the z-axis: $s_z = \dfrac{\hbar}{2}\cos 15$

(e 3) What is the probability of measuring spin-*down* along the x-axis (instead of z-axis)?

$$\Pr(|x-\rangle) = |\langle x-|\chi\rangle|^2 = \left| \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} \cos 7.5 \\ \sin 7.5\left(1/\sqrt{2}\right)(1+i) \end{pmatrix} \right|^2 = \left| \frac{\cos 7.5}{\sqrt{2}} - \frac{\sin 7.5}{2} - i\frac{\sin 7.5}{2} \right|^2$$

$$= \left(\frac{\cos 7.5}{\sqrt{2}} - \frac{\sin 7.5}{2} \right)^2 + \left(\frac{\sin 7.5}{2} \right)^2 \approx 0.408 \quad (\text{numerical values not needed for credit})$$

Or, this can be further simplified as:

$$\Pr(|x-\rangle) = \left| \frac{1}{\sqrt{2}} [1 \;\; -1] \begin{pmatrix} \cos 7.5 \\ \sin 7.5 e^{i\pi/4} \end{pmatrix} \right|^2 = \frac{1}{2}\left(\cos 7.5 - \sin 7.5 e^{+i\pi/4}\right)\left(\cos 7.5 - \sin 7.5 e^{-i\pi/4}\right)$$

$$= \frac{1}{2}\left[\cos^2 7.5 + \sin^2 7.5 - \cos 7.5 \sin 7.5\left(e^{+i\pi/4} + e^{-i\pi/4}\right)\right]$$

$$= \frac{1}{2}\left[1 - \cos 7.5 \sin 7.5\left(2\cos\frac{\pi}{4}\right)\right] = \frac{1}{2}\left[1 - \frac{\sin 15}{\sqrt{2}}\right] \approx 0.408$$

Or, we can find the probability from the angle between the spin-axis and the $+x$-axis $\equiv \alpha$:

From geometry: $\cos\alpha = \sin\theta\cos\phi$

$$\Pr(|x-\rangle) = \sin^2 \alpha/2 = \frac{1}{2}(1-\cos\alpha) = \frac{1}{2}(1-\sin\theta\cos\phi) = \frac{1}{2}\left(1 - \frac{\sin 15}{\sqrt{2}}\right), \textit{ as above}.$$

4. (15 pt) A proton is a spin-1/2 particle. It sits in a magnetic field of $\mathbf{B} = B\mathbf{e_z}$. The hamiltonian of a spin-1/2 particle in such a B-field is $-\mu_B \sigma_z B$, where σ_z is a Pauli matrix, and μ_B is a constant.

(a 4) What is the average energy due to the magnetic field, of the proton in the state $|z+\rangle$?

$$E^{z+} = \langle z+| -\mu_B \sigma_z B |z+\rangle = -\mu_B B \langle z+| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |z+\rangle = [1 \;\; 0] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\mu_B B$$

Note that the energy is negative.

(b 3) What is the average energy due to the magnetic field, of the proton in the state $|z-\rangle$?

Same magnitude, but now it's positive: $E^{z-} = +\mu_B B$

(c 4) What is the average energy due to the magnetic field, of the proton in the state $|x+\rangle$?

Since σ_z is diagonal, each component of z acts independently. The positive and negative contributions cancel, so the energy = 0. Or: $\langle x+| -\mu_B \sigma_z B |x+\rangle = \left[1/\sqrt{2} \;\; 1/\sqrt{2}\right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 0$

(d 4) The proton is in the state $|x+\rangle$ at $t = 0$. When will it next be in the state $|x+\rangle$?

$$\hbar\omega = E^{z+} - E^{z-} = -2\mu_B B \quad \Rightarrow \qquad T = \frac{2\pi}{|\omega|} = \frac{2\pi\hbar}{2\mu_B B} = \frac{\pi\hbar}{\mu_B B}$$

5. (15 pt) A particle with spin s = 1 is in state $\left|\chi\right\rangle = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{3} \\ 1/\sqrt{2} \end{pmatrix}$. It travels in the +y direction, and enters a Stern-Gerlach device which is aligned with the *z*-axis. A screen blocks the $m = -1$ state at the output of the SG.

(a 5) What is the probability that the particle leaves the SG (bypassing the screen)?

This is probability that $m = 1$ or $m = 0$: $\Pr = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2 = \frac{1}{2}$

(b 5) What is the state of the particle after it leaves the SG, and the spatial paths are recombined?

We retain only the m = 1 and m = 0 components, and normalize, which from above, requires a factor of √2:

$$\left|\chi\right\rangle = Norm\begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ -\sqrt{2/3} \\ 0 \end{pmatrix}$$

(c 5) The particle goes through a 2nd SG, tilted in the x-z plane from the 1st SG. The eigenvector for $m = 1$ *along this tilted axis* is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. What is the probability that the particle leaves the SG in the $m = 1$ state for this tilted axis?

$$\Pr\left(m_{tilted} = 1\right) = \left|\left\langle m_{tilted} = 1 \middle| \chi\right\rangle\right|^2 = \left|\begin{pmatrix} a^* & b^* & c^* \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} \\ -\sqrt{2/3} \\ 0 \end{pmatrix}\right|^2 = \left|a^*/\sqrt{3} - b^*\sqrt{2/3}\right|^2 = \left|a/\sqrt{3} - b\sqrt{2/3}\right|^2$$

6. (15 pt) Alice and Bob receive spin-entangled particles in the state $\frac{1}{2}\left|\uparrow\downarrow\right\rangle + \frac{1}{\sqrt{2}}\left|\downarrow\downarrow\right\rangle + \frac{1}{2}\left|\downarrow\uparrow\right\rangle$, given in the z-basis, where Alice's spin is given first in the ket, and Bob's 2nd.

(a 3) What is the probability that Alice measures up *and* Bob measures down?

The probability is the magnitude squared of the coefficient of the given state: $\Pr\left(\left|\uparrow\downarrow\right\rangle\right) = (1/2)^2 = 1/4$

(b 3) If Alice measures up, what is the probability Bob then measures down?

When Alice measures, the state collapses to the maximum consistent with her measurement. There is only one such collapsed possibility: Pr(Bob ↓) = 1.

(c 3) If Bob measures down, what is the probability Alice then measures up?

Similarly, the state after Bob's measurement is:

$$Norm\left(\frac{1}{2}\left|\uparrow\downarrow\right\rangle + \frac{1}{\sqrt{2}}\left|\downarrow\downarrow\right\rangle\right) = \sqrt{\frac{1}{3}}\left|\uparrow\downarrow\right\rangle + \sqrt{\frac{2}{3}}\left|\downarrow\downarrow\right\rangle. \quad \Pr\left(\left|\uparrow\downarrow\right\rangle\right) = \left(\sqrt{1/3}\right)^2 = 1/3$$

(d 3) What is the probability that Alice and Bob both measure up?

The coefficient of |↑↑> is zero, so Pr( ) = 0.

(e 3) If Bob measures down, and Alice then measures in the x-direction, what is her $\langle s_x \rangle$ over many particles?

$$From\ part\ (c),\ state = \sqrt{\frac{1}{3}}\left|\uparrow\downarrow\right\rangle + \sqrt{\frac{2}{3}}\left|\downarrow\downarrow\right\rangle. \quad Alice's\ particle\ is\ just\ \left|A\right\rangle = \sqrt{\frac{1}{3}}\left|\uparrow\right\rangle + \sqrt{\frac{2}{3}}\left|\downarrow\right\rangle$$

$$\langle s_x \rangle = \langle A | s_x | A \rangle = \begin{bmatrix}\sqrt{1/3} & \sqrt{2/3}\end{bmatrix} \frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{1/3} \\ \sqrt{2/3}\end{pmatrix} = \frac{\hbar}{2}\begin{bmatrix}\sqrt{1/3} & \sqrt{2/3}\end{bmatrix}\begin{pmatrix}\sqrt{2/3} \\ \sqrt{1/3}\end{pmatrix} = \frac{\hbar}{2}\left(\sqrt{2/9} + \sqrt{2/9}\right)$$

$$= \hbar\frac{\sqrt{2}}{3}$$