

# Physics 161: Black Holes: Lecture 4: 11 Jan 2010

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## 4 General and Special Relativity

GR is based upon **the equivalence principle: All local inertial frames are equivalent for performance of all physical experiments.** Remember that a **local inertial reference frame is one that is free falling and non-rotating**, where particles move at constant velocities in straight lines (or stay still). A very important point is that Special Relativity is how spacetime behaves in any inertial frame. This is equally true for someone out in deep space far from anything, for someone orbiting the Earth in a spaceship, or for someone falling in an elevator with broken cables. All three someones are in inertial reference frames and would find the same result for any experiment they did; that is they could not tell which of the situations they were in, without looking outside (looking outside makes the frame bigger, i.e. non-local). The influence of the Earth's gravity on the guy falling in the elevator is not measurable by local experiments.

The metric of Special Relativity (SR) is the Minkowski metric  $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ , and GR is an extension of SR. So we start by understanding the SR metric. In SR the relations between any two local inertial frames are the Lorentz transformations which you have seen before. These, and all of SR, can be derived from two postulates:

1. The principle of relativity: results of experiments don't depend on velocity of lab.
2. All inertial frames measure the same speed of light which is exactly  $c = 299792458$  m/s! This postulate is the same as requiring the invariant interval  $ds$  to be the same in every frame.

These postulates can be used to derive the Lorentz transformations.

### 4.1 Spacetime diagrams

A very nice tool are the spacetime diagrams. They will be indispensable when we try and go inside a black hole, so it is worthwhile trying to understand them.

#### Figure: Intro to Spacetime diagram

Points:

- Axes are  $x$  vs.  $t$ . Points are events, lines represent observers moving through spacetime (worldlines) or extended objects (if spacelike).
- The slope of an observer's worldline is  $dt/dx = 1/v$ .

- With  $c = 1$ , an observer moving at the speed of light is a  $45^\circ$  line. Nothing can move at a shallower slope, since that would mean  $v > c$ . Thus only events within the “lightcone” can be casually connected. [We suppress the  $y$  and  $z$  directions; if we include  $y$  then it is a cone.]
- Straight lines represent inertial observers (each related by a Lorentz transformation.) A line straight up is someone standing still in the lab frame.
- Curved lines represent accelerated observers.

## 4.2 More special relativity: how time and space appears to other observers: Lorentz transformation

Now let’s use spacetime diagrams to understand some of the weirdness of special relativity. Suppose observer  $O$  uses  $(t, x)$  and is in the lab, while observer  $O'$  uses  $(t', x')$  and is moving in the  $x$ -direction with velocity  $v$ .

### Fig: $O'$ moving at $v$ in $O$ coordinates

Where do the  $t'$  and  $x'$  coordinate axes go in this diagram? Well, the worldline of  $O'$  is their  $t'$  axis, since this is  $x' = 0$  to them (and also  $y' = z' = 0$ ). Where is the  $x'$  axis? To find this consider  $O'$  spacetime diagram from  $O'$ ’s point of view.

### Fig: $O'$ spacetime diagram in $O'$ coordinates: reflecting light

From  $O'$  view  $t'$  axis goes straight up and  $x'$  axis is at right angles. Consider a photon emitted at  $(t', x') = (-a, 0)$  in the  $+x'$  direction. It reaches the  $x'$  axis at  $t' = 0$ , and then is reflected back (by a mirror). It returns to the  $t'$  axis ( $x' = 0$ ) at  $(t', x') = (a, 0)$ . Consider other photons emitted at different times,  $-b$ , etc. and we see that we could *define* the  $x'$  axis as the locus of points that will reflect photons emitted at  $t' = -a$ , to reach back at  $t' = a$ .

This should be a frame-independent definition, so we can find the  $x'$  axis in the  $O$  frame using it.

### Fig: $t'$ and $x'$ axes in $O$ coordinates: reflecting light

To do this we use postulate two: light travels at  $c = 1$  in all frames, that is, the light pulses  $O'$  uses to find the  $x'$  axis travel at  $45^\circ$  also in the  $O$  spacetime diagram. We see that the  $x'$  axis is *not* the same as the  $x$  axis! So events which are simultaneous to  $O'$  (i.e. happen at the same values of  $t'$ , e.g.  $t' = 0$  on the  $x'$  axis), are not simultaneous to  $O$  (don’t happen at same values of  $t$ , i.e. are not parallel to the  $x$  axis.) This is the famous failure of simultaneity in SR, and is made very obvious in spacetime diagrams.

With some work, from the above postulates one can derive the famous **Lorentz transformations**:

$$\begin{aligned} t &= \gamma(t' + \frac{v}{c^2}x') \\ x &= \gamma(vt' + x') \\ y &= y' \\ z &= z', \end{aligned}$$

where the Lorentz factor is the greek letter gamma:

$$\gamma = 1/\sqrt{1 - v^2/c^2}.$$

Note that with  $c = 1$  these simplify:

$$t = \gamma(t' + vx')$$

$$x = \gamma(vt' + x')$$

$$y = y'; z = z',$$

with  $\gamma = 1/\sqrt{1-v^2}$ .

One can easily find the inverse transforms, just by setting  $v \rightarrow -v$  in the above: **Inverse Lorentz transformation:**

$$t' = \gamma(t - vx)$$

$$x' = \gamma(-vt + x)$$

In the homework you will use these to prove that  $\Delta s$  between two events is the same in every frame.

### 4.3 Calibrating the axes in a spacetime diagram: hyperbolas=circles!

#### Fig: Hyperbolas on space time diagram

How does one calibrate the axes in a spacetime diagram? Consider the equation:  $t^2 - x^2 = a^2$ . All the points on this hyperbola have the same invariant “distance” from  $(0,0)$ .  $\Delta s^2 = -\Delta t^2 + \Delta x^2 = -a^2$ ; e.g.  $x = 0$  and  $t = a$ , etc. Thus this hyperbola is the equivalent of a circle in normal Euclidian geometry. The proper time is the time that someone traveling along the line would measure on their wristwatch. So no matter what the speed, all travelers from the origin to the hyperbola take the same time to get there (i.e. they age the same amount,  $a$ ). Note that  $-a^2$  is the invariant interval, and since  $ds^2 = -a^2 < 0$  this hyperbola is for timelike separations.

Similarly one can plot the equation:  $t^2 - x^2 = -b^2$ , at get the sideways hyperbola which represents spacelike separations. These points all have the same proper length from the origin.

#### Fig: Events A and B on space time diagram

To calibrate the axes, we consider two events, A and B. A lies at  $(t, x) = (1, 0)$  on the  $O t$  axis, while event B lies at  $(t', x') = (t', 0)$  on the  $O' t'$  axis. We pick B to lie along the hyperbola so the invariant interval (proper time) from the origin is the same to both events A and B. Since the interval  $\Delta s$  is invariant (same in all frames), and event A has  $t = 1, x = 0$ , we have  $\Delta s^2 = t^2 - x^2 = 1$ . Therefore in moving frame,  $t'^2 - x'^2 = -1$  also. Event B occurs at  $x' = 0$ , which implies  $t' = 1$ . But on the spacetime diagram, the distance from the  $(0,0)$  to event B looks farther than the distance from  $(0,0)$  to event A! The point here is that one can't use Euclidian intuition about distances on spacetime diagrams: It is Minkowski geometry. The proper distance is  $\Delta s^2 = -\Delta t^2 + \Delta x^2$ , *not*  $\Delta t^2 + \Delta x^2$  as we are used to. Now you see the importance of the minus signs in the spacetime metric. They change the geometry from Euclidian to Minkowski.

In the above, the moving frame observer,  $O'$ , says  $t' = 1$ , but the lab frame says the clock tick happened at a longer time. (Just read off the  $t$  time from the  $t$  axis!)

This says that the moving person's time runs slowly, when observed from the lab frame.

Finally, note that on a space time diagram we can easily see which events are separated by time-like intervals and which are separated by space-like intervals. Thus we can see which events can and cannot be causally connected. **The 45° lines are called the lightcone.** If we add say the y dimension you see why it is called a cone, but if we add the y and z dimensions it is actually a cone in 4-D! The cone opening towards larger time is called the future lightcone and contains all future events that can be influenced by it. The cone opening towards the past is call the past lightcone and contains all the events that could have influenced the event at the apex of the cones. You can draw a lightcone around any event and see which events can influence which other events.