# Physics 161: Black Holes: Lecture 28/29: 10/12 Mar 2010

Professor: Kim Griest

## 28 Gravitational Waves

### 28.1 Introduction

Newton's law of gravity has a problem. Consider two masses $M_1$ and $M_2$ separated by a long distance. According to Newton, the force between them is $F = GM_1M_2/r^2$. Now suppose you move the first mass closer. Newton's law predicts that the force on the 2nd mass changes, and in fact changes instantly. But that can't happen. Things cannot communicate faster than the speed of light without violating causality. Newton's thus law violates causality. This is a problem, which luckily Einstein's theory of GR fixes. When the first mass is moved, the metric of spacetime is changed, and the solution to Einstein's field equations shows that this change in metric propagates outward at the speed of light. It is actually very similar to electricity and magnetism, where if one moves a charge it causes a change electric field that propagates outward at $c$. This is called electromagnetic radiation. Thus one expects that GR has something like gravitational radiation, aka gravity waves in it.

In order find these, we can proceed in the same way as you would in E&M. In E&M you write down Maxwell's equations and set all the sources, e.g. charges and currents to zero. This gives the wave equation $(\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2)\vec{E} = 0$, which has solution $E_i = A_i \exp(\omega t - \vec{k}\cdot\vec{x})$; that is a sine or cosine traveling wave moving in direction $\vec{k}$, with polarization $A_i$. Plugging this solution into the equation gives $\omega = \pm ck$. Since the speed of a wave is $v = \omega/k$, this implies electromagnetic radiation travels at the speed of $v = c$, the speed of light.

### 28.2 Linearized Weak Field GR

We would like to do the same thing with Einstein's field equations, but as we have seen these are substantially more complicated:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where we set the cosmological constant to zero. For gravitational radiation, we want to set all 16 terms in the stress-energy tensor $T_{\mu\nu}$ to zero, and then untangle the complicated differential equations implied in $G_{\mu\nu}$ to solve for the metric $g_{\mu\nu}$, where the line element we have been using is $ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2$. This is hard, but can be made easier if we employ the "weak field" limit. This is an example of perturbation theory, where we assume that whatever answer we get for $g_{\mu\nu}$ will be very close to the flat space Minkowski metric $\eta_{\mu\nu}$, where remember $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and all other elements of $\eta_{\mu\nu}$ are zero. Thus we start by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu} \ll 1$. We expand out the field equations, dropping any terms of order $h^2$ or higher. Remarkably this gives

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\mu\nu} = -\frac{16\pi G}{c^2}T^{\mu\nu},$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$, is the "transverse-traceless weak field metric", and $h = h^{\mu}_{\mu}$ is the trace.

## 28.3 Connection with Newton

Before using this equation for gravitational waves we can make a connection with Newton by noting that for normal slowly moving matter the only significant element of the stress-energy tensor is $T^{00} = \rho$, the mass density. Thus we can solve this equation using only the 00 component. If we set up a mass density and look for a static solution, we can also set the time derivative to zero, giving

$$\nabla^2\bar{h}^{00} = -16\pi G\rho.$$

This is almost exactly the equation for the Newtonian potential $\nabla^2\phi = 4\pi G\rho$, where in the Newtonian case for a spherical mass $M$, the gravitational potential would be $\phi = -GM/r$. Thus we can think of $h$ as the gravitational potential with $\bar{h}^{00} = -4\phi$, $h^{00} = -2\phi$, $h^{11} = h^{22} = h^{33} = -2\phi$, and $h = -4\phi$. Plugging this into the line element (metric) we get $ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$, where we used our perturbation expansion, $g^{00} = -1 + h^{00}$, $g^{11} = 1 + h^{11}$, etc.

Finally, compare this to the Schwarzschild metric, for simplicity paying attention only to the time and radial components, $dr^2 \sim dx^2+dy^2+dz^2$. In the weak field limit one can Taylor expand the $(1-2GM/r)^{-1}$ term in the Schwarzschild metric to find to find $ds^2 \approx (-1+2GM/r)dt^2 + (1+2GM/r)dr^2 + ...$, precisely the weak field metric found above with $\phi$ substituted. Thus the connection between the Newtonian potential and the metric becomes more clear.

## 28.4 Gravitational Waves

Now set $T^{\mu\nu} = 0$ in the weak field Einstein equations. This gives

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\mu\nu} = 0,$$

again a wave equation for the 16 components of $\bar{h}^{\mu\nu}$. The solution is again sine and cosine traveling waves

$$\bar{h}^{\mu\nu} = A^{\mu\nu}\exp(\omega t - \vec{k}\cdot\vec{x}),$$

where now the "polarization", $A^{\mu\nu}$, of the gravity wave has more components. Again we see $\omega = \pm k$, or the speed of gravity waves is 1 (that is $c$). It is worth discussing the polarization since it differs from electromagnetism. Recall that an electromagnetic wave traveling in the $z$ direction is transverse, meaning it can have an "x" component or a "y" component, but no "z" component. This is set by the gauge condition in electromagnetism. A gravity wave has more components and it's gauge condition means that it is quadrapolar. Traveling in the z direction it has two possible polarizations, one called + polarization with $A^{xx} = -A^{yy}$ and all other components zero, and the other, called $\times$, with $A^{xy} = A^{yx}$ and all other components zero (note I am calling $A^{11}$ as $A^{xx}$, etc.) This polarization has a big effect on what gravity waves do.

For example, consider a gravity wave traveling in the z direction with + polarization. The spatial part of the metric will be something like: $ds^2 = (1 + h^{11})dx^2 + (1 + h^{22})dy^2 + (1 + h^{33})dz^2$, where we should substitute $h^{11} = A^{11}\sin(\omega t)$, $h^{22} = h^{yy} = A^{yy}\sin(\omega t)$, $h^{33} = A^{zz}\sin(\omega t)$, and we assume we are at $z = 0$, so we can set $\vec{k} \cdot \vec{x} = 0$. This gives: $ds^2 = (1 + h\sin(\omega t))dx^2 + (1 - h\sin(\omega t))dy^2 + dz^2 - ...dt^2$. This means that there will be no change in the $z$ direction, and opposite sinusoidal motion in the $x$ and $y$ directions. What will this do?

Imagine a circular ring of test particles in the x-y plane. As the gravity wave comes through what will happen? What will the forces on the particles be? Zero of course! In GR gravity is not a force! In fact, the coordinate positions ($x$, $y$, $z$) will not change at all. But using the metric above we see that the distances between the particles will move! This is the effect of the gravity wave passing by. When the $\sin(\omega t)$ in the x-direction is maximum ($\omega t = \pi/2$), it will be at a minimum in the y-direction. Thus the circle of test particles will alternatively squeeze and stretch in the x and y directions. You can work out the $\times$ polarization also, it is similar except the squeezing and stretching at a $45^0$ angle.

**Figure: Effect of passing gravitational wave on ring of test particles**

So to detect gravity waves all you have to do is measure the distances between test masses. They should move in the odd pattern just described when a gravity wave comes by. Do we expect gravity waves to exist? Yes of course. As we said at the beginning there has to be gravity waves whenever masses move, just as in electromagnetism there has to be electromagnetic waves whenever charges move. But be careful. A charge moving at uniform velocity does cause a change in the electric field to propagate outward, but does not radiate electromagnetic wave waves that carry energy. It takes an accelerating charge to create electromagnetic waves. Likewise it takes an accelerating movement of mass to create a gravitational wave. In fact, due to the quadrapolar nature of the polarization, it takes a quadrapolar motion to do it. For example, an expanding spherical shell of mass has no quadrapole moment and therefore does not emit gravitational waves. However, if two masses move back and forth (say on a spring) or orbit around each other (like two stars) then there is a quadrapole moment and there is gravitational radiation.

The formula for the strain (i.e. $h$, i.e. change in the metric) when two object of mass $M$ orbit each other is roughly:

$$A \sim \frac{GMl_0^2\omega^2}{rc^4},$$

where $l_0$ is the rough size of the orbit, $\omega$ is the angular velocity of the orbit, and $r$ is your distance from the system. We can make a good rule of thumb by noting that Schwarzschild radius $GM/c^2 \approx r_S$, and the speed of orbit is roughly $v \approx l_0\omega$. so

$$A \approx \left(\frac{r_S}{r}\right)\left(\frac{v}{c}\right)^2.$$

Consider then a barbell with two 100 kg masses separated by 1 m, spinning at 10 m/s. What is the size of the metric distortion caused by the resulting gravitational wave? Using $r_S = 3km(M/M_\odot)$,

$$h \sim A \sim (\frac{3000\text{m}}{1\text{m}})(\frac{100\text{kg}}{2\times 10^{30}\text{kg}})(\frac{10m/s}{3\times 10^8 m/s})^2 \sim 10^{-40}.$$

Recall this $A$ is the fractional change in distance between two test particles sitting 1 m from the barbells. This is an absolutely tiny distortion, much much less than the radius of an atomic nucleus. So this is not measurable.

## 28.5 Detecting Gravitational Waves

Can one think of anything that would give rise to to a measurable gravitational wave? Well, in the equation above, we need to increase both the masses and speeds involved. How about two neutron stars (masses 1.4 $M_\odot$) orbiting very close to each other? In 1974, Hulse and Taylor discovered a binary pulsar, PSR 1913+16. This is system of two neutrons stars orbiting other with a period of 7.75 hours. The system is about 6.4 kpc (21,000 light years) away from us, so plugging these numbers in you find $A \sim 10^{-23}$, a much bigger value. But over one meter this is a change in distance of less than 100 million times the radius of a proton, unmeasurable with current equipment. Still the 1993 Nobel prize was given to Hulse and Taylor for their discovery of gravitational radiation using this system. Why?

Well the beauty of this system is that one of the neutron stars is a pulsar. It is rapidly spinning and beaming us with a period of $P = 0.059029997929613(7)$ seconds. This is an incredibly accurately measured period, and for a long time this system was more stable and accurate than any atomic clock here on earth. Using this information and the measured orbital period everything about this system could be understood. It all checked out using Einstein gravity, It was easy to also detect that this well measured period changes with time in a regular manner, at a rate of about, $dP/dt = 8.62713 \times 10^{-18}$. Why does the period change? Something is bleeding energy from the system. If one plots the period over a period of years it falls in exactly the manner predicted by GR, but only if one includes the energy lost from gravitational radiation. Thus this was an indirect measurement of gravitation wave emission. To this day, this system (and others like it) are the only "detections" we have of gravitational radiation.

**Figure: Orbital decay of PSR B1913+16.The data points indicate the observed change in the epoch of periastron with date while the parabola illustrates the theoretically expected change in epoch according to general relativity.**

Is there hope to ever directly detect a gravitational wave? Well we need to be able to measure strains of around $10^{-23}$. Of course if objects were closer than PSR 1913+16 or involved heavier objects (e.g. super massive black holes weighing billions of solar masses) then larger values of $h$ might be produced. The current leading experiment is called LIGO, the Laser Interferometer Gravitational-Wave Observatory. It is two large Michelson interferometers each about 4 km long. One interferometer is in Lousiana and the other is in Washington state. Each interferometer has large test masses carefully hung in vacuum pipes 4 km apart in an L shape. There are mirrors attached to the test masses and laser systems that carefully measure the distances between these test masses. If a gravity wave comes from outer space onto the detector, one arm of the interferometer will stretch while the other shrinks. This shrinking/stretching shape changing will reoccur with the frequency of the gravity wave. For instance, for the binary pulsar, the period will be 7.5 hours. However, LIGO cannot detect this kind of period; its sensitivity is only to waves with frequencies between 30 Hz and 7000 Hz. There are very few astronomical sources of strong gravity waves that emit at these frequencies, so in fact, there really is almost nothing that LIGO can detect. LIGO started in 2002 and still has not detected anything except noise. In fact, when trying to measure distances to a tiny fraction of the radius of the proton, there are innumerable sources of noise: thermal motion, ground motion, traffic, logging, ocean waves, etc, etc. etc. This is why there are two interferometers separated by about 3000 km. A gravity wave from space will hit both detectors, while noise sources will affect one but not the other. Only signals that appear in both detectors are being considered as real gravity waves. However, there is some hope for LIGO since they are upgrading it. This will make it sensitive to strains about 10 times smaller, and there are actual astronomical sources of gravity waves in the frequency band that might be detected.

Finally, there is an exciting proposed space mission called LISA, the Laser Interferometer Space

Antenna, that if launched, would give huge numbers of detections. Here the interferometer arms would be around 5 million km! This long distance moves the range of frequency sensitivity down to between $10^{-5}$ Hz and 0.1 Hz, just in the range of binary pulsars, orbiting black holes and other likely astronomical phenomena. Here the interferometer is made of 3 free flying spacecraft orbiting the Sun (not the Earth!) and the lasers will be measuring the distances between the test masses made of pure gold and platinum to about 20 picometers, for a strain sensitivity of around $10^{-23}$ as required.