Designing a Voltmeter

$\chi^2$ testing

Lecture # 8
Physics 2BL
Summer 2010
Outline

• Experiment 4 – electrical forces
• Torsional pendulum
• Review of procedure
• Uncertainties
Purpose

• Design a means to measure electrical voltage through force exerted on charged object

• Use Torsional pendulum
• Balance forces, balance torques
Experiment 4

Construct a device to measure the absolute value of a voltage through the measurement of a force

The actual measurements you will make will be of mass, distance, and time but the result will be a measurement of an electric potential in Volts

- Measure voltage difference with a standard meter
- Measure force by deflection
- We can calibrate the voltmeter

Measure force and voltage
The Four Experiments

• **Determine the average density of the earth**
  – Measure simple things like lengths and times
  – Learn to estimate and propagate errors
• **Non-Destructive measurements of densities, structure**
  – Measure moments of inertia
  – Use repeated measurements to reduce random errors
• **Test model for damping; Construct and tune a shock absorber**
  – Damping model based on simple assumption
  – Adjust performance of a mechanical system
  – Demonstrate critical damping of your shock absorber
  – Does model work? Under what conditions? If needed, what more needs to be considered?
• **Measure coulomb force and calibrate a voltmeter.**
  – Reduce systematic errors in a precise measurement.
Basic Equations

\[ F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} \quad \text{Force between point charges} \]
\[ \varepsilon_0 = 8.85 \times 10^{-12} \quad \frac{F}{m} \quad \text{Permittivity constant} \]

\[ E = \frac{F}{Q_2} = \frac{Q_1}{4\pi \varepsilon_0 r^2} \quad \text{Electric field from a point charge} \ Q_1 \]
\[ \text{Coulomb force acting on a unit charge} \]

\[ V = \frac{1}{Q_2} \int_{r}^{\infty} F dr = \frac{Q_1}{4\pi \varepsilon_0 r} \quad \text{Voltage - potential energy per unit charge} \]

\[ \Delta V = \int_{r_1}^{r_2} Edr = -\frac{Q_1}{4\pi \varepsilon_0 r_1} + \frac{Q_2}{4\pi \varepsilon_0 r_2} \quad \text{Voltage difference} \]
Parallel Plate Capacitor

We suggest the use of a parallel plate capacitor rather than charged spheres.

\[ E = \frac{Q}{A\varepsilon_0} \]  
from Gauss’s Law

\[ V = Ed = \frac{Qd}{A\varepsilon_0} \]  
voltage difference

\[ F = \frac{1}{2} EQ = \frac{1}{2} \frac{Q^2}{A\varepsilon_0} = \frac{1}{2} \frac{A\varepsilon_0}{d^2} V^2 \]  
the force

\[ F = \frac{1}{2} \left( A = 3 \text{ cm}^2 \right) \left( \varepsilon_0 = 8.8 \times 10^{-12} \frac{F}{\text{m}} \right) \left( V = 1000 \text{ V} \right)^2 = 1.2 \times 10^{-3} \text{ N} \]

The weight of 0.1 g.
Calibrate Voltmeter

- Set up the apparatus.
- Keep table dry.
- Make the plates parallel for spacer in contact.
- Measure the spacer.
- Measure $\kappa$.
- Find Voltage that just causes plates to move apart.
- Try calibration at about 1000 Volts.
- Now get several measurements at lower voltage.
- Water must be stable.
- Move slowly.
- Protect your apparatus from air currents.
- Estimate errors
Voltmeter Apparatus

- Angle/torque adjuster
- Torsion fiber
- Hanger
- Power supply (battery)
- Mirror
- Damping bath
- Capacitor plates
Experimental Technique

- Because of the small forces involved, the apparatus is very sensitive to
  - flow in the water
  - air currents
  - vibrations
- We can get these to a minimum but we can’t eliminate them
Measure $\kappa$ using Torsional Pendulum

$$F = \kappa \theta / l$$

- $l$ - Distance from the suspension to the disk is measured with a ruler
- $\theta$ - Deflection angle is measured with a protractor

How do we measure the torsion constant $\kappa$?

Torsional oscillations

$$T = 2\pi \sqrt{\frac{I}{\kappa}}$$

$$\kappa = \left( \frac{2\pi}{T} \right)^2 I$$

$I$ - Moment of inertia

$$\varepsilon_\kappa = \sqrt{(\varepsilon_I)^2 + (2\varepsilon_T)^2}$$

$$\sigma_\kappa = \kappa \ast (\varepsilon_\kappa)$$
Moment of Inertia

\[ I = \frac{1}{3} ml^2 + (m_1 + m_2)l^2 + (m_1 + m_2)\frac{R^2}{4} \]

rod parallel axis theorem disks - CM

\[ \sigma_I = \sqrt{[\frac{\delta I}{\delta m} \sigma_m]^2 + [\frac{\delta I}{\delta m_1} \sigma_{m_1}]^2 + [\frac{\delta I}{\delta m_2} \sigma_{m_2}]^2 + [\frac{\delta I}{\delta l} \sigma_l]^2 + [\frac{\delta I}{\delta R} \sigma_r]^2} \]
Error in Moment of Inertia

\[ I = \frac{1}{3}(ml^2) + (m_1 + m_2) l^2 + \frac{(m_1 + m_2)R^2}{4} \]

\[ \sigma_I = \sqrt{\left[ (\frac{\delta I}{\delta m})\sigma_m \right]^2 + \left[ (\frac{\delta I}{\delta m_1})\sigma_{m_1} \right]^2 + \left[ (\frac{\delta I}{\delta m_2})\sigma_{m_2} \right]^2 + \left[ (\frac{\delta I}{\delta l})\sigma_l \right]^2 + \left[ (\frac{\delta I}{\delta R})\sigma_r \right]^2} \]

\[ \frac{\delta I}{\delta m} = \frac{1}{3}(l^2) \]

\[ \frac{\delta I}{\delta m_1} = l^2 + \frac{R^4}{4} = \frac{\delta I}{\delta m_2} \]

\[ \frac{\delta I}{\delta l} = \frac{2}{3}(ml) + 2(m_1 + m_2)l \]

\[ \frac{\delta I}{\delta R} = \frac{1}{2}(m_1 + m_2)R \]
Capacitor – Electrical Force

Damping water bath

Hanger

Brass disk capacitor plate

Aluminum capacitor plate
Angular
Torsion

- Electric connector
- Rotating support for torsion fiber
- Protractor
- Lever to adjust angle
- Torsion fiber
Equilibrium Positions

\[ F = \frac{1}{2} \frac{A \varepsilon_0}{d^2} V^2 \]  
\text{electrostatic attraction}

\[ Ft \]  
\text{torque resulting from the electrostatic force}

\[ k \theta \]  
\text{torque resulting from the fiber}

\text{hold separation between the capacitor plates fixed as the voltage between them is increased by twisting the top end of the fiber}

\[ V = d \sqrt{\frac{2k \theta}{lA \varepsilon_0}} \]
Error Propagation

\[ V = d \sqrt{\frac{2\kappa\theta}{lA\varepsilon_0}} \]

\[ \varepsilon_V = \sqrt{(\varepsilon_d)^2 + (\varepsilon_k/2)^2 + (\varepsilon_\theta/2)^2 + (\varepsilon_A/2)^2} \]

\[ \sigma_V = V \times (\varepsilon_V) \]
Assembling & Testing the calibrator - step by step (i)

1. Attach mirror to the support beam and set up the laser to project a spot on the wall.

2. Bring the torsion pendulum to equilibrium so that it is not moving.
   - Mark the position of the laser spot on the wall with a piece of tape.

3. Measure the period of the oscillations by watching the spot on the wall and then calculate the torsion constant of the fiber.

4. Add damping to the system to limit unwanted oscillations (water bath)

5. Bring the pendulum to equilibrium and place the fixed capacitor plate parallel to the moveable plate, just barely touching the insulating dot.
Assembling & Testing the calibrator - step by step (ii)

6. Apply 1000 volts across the capacitor.
   - The plates will clamp together.
   - Read “initial” angle

7. Apply a known torque (in the direction to pull the plates apart) by rotating the torque adjustment lever, until plates separate
   - Read “final” angle
   - Compute Angle difference - $\Delta \theta$

8. Compute the voltage using:

   $$ V = d \sqrt{\frac{2k\Delta \theta}{lA\varepsilon_0}} $$

9. Repeat several times (at different voltages 600-1000 V)
   - 5 voltages
   - Do above procedure at least 3 times
Analysis

Make a graph of your data where:
  • x-axis is the voltage read from the power supply (600-1000V)
  • y-axis is the calculated voltage from the torsional pendulum

Fit to straight line
Calculate $\chi^2$
Discuss goodness of fit
Calculate probability of result.
\( \chi^2 \) Testing
(Taylor Chapter 12)

- You take \( N \) measurements of some parameter \( x \) which you believe should be distributed in a certain way (e.g., based on some hypothesis).
- You divide them into \( n \) bins \((k=1,2,...,n)\) and count the number of observations that fall into each bin \((O_k)\).
- You also calculate the expected number of measurements \((E_k)\), in the same bins, based on some hypothesis.
- Calculate:
\[
\chi^2 = \sum_{i=1}^{n} \frac{(O_k - E_k)^2}{E_k}
\]
- If \( \chi^2 < n \), then the agreement between the observed and expected distributions is acceptable.
- If \( \chi^2 \gg n \), there is significant disagreement.
Degrees of Freedom

- Number of degrees of freedom, \( d \), is the number of observations, \( O_k \), minus the number of parameters computed from the data and used in the calculation.

\[ d = n - c, \]

- Where \( c \) is the number of parameters that were calculated in order to compute the expected numbers, \( E_k \).
- It can be shown that the expected average value of \( \chi^2 \) is \( d \).

- Therefore, we define “reduced chi-squared”:

\[ \tilde{\chi}^2 = \frac{\chi^2}{d} \]

- If the reduced chi-squared is <1, there is no reason to doubt the expected distribution.
Fitting Summary

- You have a set of measurements and a hypothesis that relates them.
- The hypothesis has some unknown parameters that you want to determine.
- You “fit” for the parameters by maximizing the odds of all measurements being consistent with your hypothesis.
- Evaluate your fit based on the goodness of fit.
Example - Dice

Die is tossed 600 times

Expectation: each face has same likelihood of showing up

Verification of expectation by computing the $\chi^2$

<table>
<thead>
<tr>
<th>Y</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>obs</td>
<td>91</td>
<td>137</td>
<td>111</td>
<td>87</td>
<td>80</td>
<td>94</td>
</tr>
<tr>
<td>exp</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

$\Delta^2$ 81 1369 121 169 400 36

$\chi^2_i$ 0.81 13.7 1.21 1.69 4.0 0.36

Total $\chi^2$ 21.76

$n_{ dof}$ 5

Reduced $\chi^2$ 4.35

This term is the squared difference between observation and expectation.

In computation of $\chi^2$ the $\Delta^2$ term is divided by expectation. $\sigma$ is square root of expectation ($E_y = \sigma_y^2$)
Application of $\chi^2$ – Use of Table D

Just calculated:

<table>
<thead>
<tr>
<th>Total $\chi^2$</th>
<th>21.77</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{dof}$</td>
<td>5</td>
</tr>
<tr>
<td>Reduced $\tilde{\chi}^2$</td>
<td>4.35</td>
</tr>
</tbody>
</table>

Die is loaded at 99.9% Confidence Level
\( \chi^2 \) Test for a Fit

- We have used \( \chi^2 \) minimization to fit data.
- We can also use the value of \( \chi^2 \) to determine if the data fit the hypothesis.
- On average, the \( \chi^2 \) value is about one per degree of freedom.
- The number of degrees of freedom is the number of measurements minus the number of fit parameters.
- We will use the \( \chi^2 \) per degree of freedom to compute a probability that the data are consistent with the hypothesis. (table D)
- This probability of \( \chi^2 \) is like the confidence level.

\[
\chi^2 = \sum_{i=1}^{n} \frac{(y_i - f(x_i))^2}{\sigma_y^2}
\]

\[
\langle \chi^2 \rangle = n_{\text{d.o.f.}}
\]

\[
n_{\text{d.o.f.}} = n_{\text{data}} - n_{\text{parameters}}
\]

\[
\tilde{\chi}^2 = \frac{\chi^2}{n_{\text{d.o.f.}}}
\]

If \( \text{Prob}(\tilde{\chi}^2 > \chi^2_\circ) \) is less than 5\% - disagreement “significant”
If \( \text{Prob}(\tilde{\chi}^2 > \chi^2_\circ) \) is less than 1\% - disagreement “highly significant”
Experimental Hints

- Because of the small forces involved, the apparatus is very sensitive to:
  - flow in the water
  - air currents
  - vibrations

- We can get these to a minimum but we can’t eliminate them

  Water must be stable.
  Move slowly.
  Protect your apparatus from air currents.
  And your partners...
Reminder

• Experiment 4 – care in measurements
• No final
• No more lectures