Physics 4E Quiz 3 Solutions Spring 2010

Problem 1

Wave function is: $\psi(x) = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L}$; probability is $(x_1 = 3.5\text{Å}, x_2 = 4.5\text{Å})$

$$P = \int_{x_1}^{x_2} dx\, |\psi(x)|^2$$. We can approximate it by taking the value of $\psi(x)$ at center and multiplying by width of interval, i.e.

$$P \approx |\psi(\tfrac{x_1+x_2}{2})|^2 \cdot (x_2 - x_1) = \frac{2}{L}\sin\left(\frac{\pi \cdot 4\text{Å}}{8\text{Å}}\right) \cdot (1\text{Å}) = \frac{2}{8\text{Å}} \cdot 1\text{Å} = 0.25$$

So $\boxed{P \approx 0.25.}$ Classically, $\boxed{P = \frac{\Delta x}{L} = \frac{1\text{Å}}{8\text{Å}} = 0.125}$, half as large.

(b) Since $\psi(x)$ decreases away from the center, the exact answer to (a) is slightly smaller.

(c) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$; $\langle p\rangle = 0$ by symmetry.

$$p = \frac{\hbar}{i}\frac{d}{dx}, \quad p^2 = -\hbar^2\frac{d^2}{dx^2}, \quad p^2\psi(x) = \frac{\hbar^2\pi^2}{L^2}\psi(x) \Rightarrow$$

$$\Rightarrow \langle p^2\rangle = \frac{\hbar^2\pi^2}{L^2}\underbrace{\int dx\, |\psi(x)|^2}_{1} = \frac{\hbar^2\pi^2}{L^2} \Rightarrow$$

$$\Rightarrow \Delta p = \frac{\hbar\pi}{L} = \frac{\hbar c \cdot \pi}{L}/c = \frac{1973\, \text{eV}\cancel{\text{Å}} \cdot \pi}{8\,\cancel{\text{Å}}}/c = \boxed{775\, \text{eV}/c}$$

(d) $$P = \frac{2}{L}\int_{x_1}^{x_2} dx \sin^2\frac{\pi x}{L} = \frac{2}{\cancel{L}}\frac{\cancel{L}}{\pi}\int_{\frac{\pi}{L}x_1}^{\pi/L\, x_2} du \sin^2 u = \left(\text{use } \sin^2 u = \frac{1-\cos 2u}{2}\right)$$

$$= \frac{2}{\pi}\left[\frac{1}{2} - \frac{\sin 2u}{4}\right]_{\frac{\pi}{L}x_1}^{\pi/L\, x_2} = \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{\pi}}{L}\cdot(x_2 - x_1) - \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{4}_2}\cdot\left(\sin\frac{2\pi}{L}x_2 - \sin\frac{2\pi}{L}x_1\right) =$$

$$= 0.125 - \frac{1}{2\pi}\left(\sin\left(\pi\cdot\frac{4.5}{4}\right) - \sin\left(\pi\cdot\frac{3.5}{4}\right)\right) = 0.125 + 0.1218$$

$$\boxed{= 0.2468}$$

Problem 2

$$E_0 = \frac{\hbar\omega}{2} = 3\,eV \quad ; \quad E_n = \hbar\omega(n+\tfrac{1}{2})$$

Photons emitted: $\frac{hc}{\lambda} = E_{n+1} - E_n = \hbar\omega \Rightarrow \lambda = \frac{hc}{\hbar\omega}$

$$\hbar\omega = 6\,eV \Rightarrow \lambda = \frac{12{,}400\,eV\mathring{A}}{6\,eV} \Rightarrow \boxed{\lambda = 2067\,\mathring{A}} \text{ (a)}$$

(b)

$$E_0 = \frac{1}{2} m\omega^2 A^2 \Rightarrow A^2 = \frac{2E_0}{m\omega^2} = \frac{\hbar\omega}{m\omega^2} = \frac{\hbar}{m\omega}$$

From $E_0 = \frac{\hbar\omega}{2} \Rightarrow \omega = \frac{2E_0}{\hbar} \Rightarrow A^2 = \frac{\hbar\,\hbar}{m \cdot 2E_0} = \frac{\hbar^2}{2mE_0} =$

$$= \frac{3.81\,eV\mathring{A}^2}{3eV} \Rightarrow \boxed{A = 1.127\,\mathring{A}}$$

(c) $$\psi_0(x) = C_0\, e^{-\frac{m\omega x^2}{2\hbar}} = C_0\, e^{-\frac{x^2}{2A^2}} \Rightarrow \psi_0(A) = C_0 e^{-\frac{1}{2}}$$

$$\frac{P(x=0)}{P(x=A)} = \frac{|\psi_0(0)|^2}{|\psi_0(A)|^2} = \frac{C_0^2}{C_0^2(e^{-1/2})^2} = e = \boxed{2.72}$$

electron in ground state is 2.72 times more likely to be at $x=0$ than at $x=A$. But there are two classical turning points, $x = A$ and $x = -A$.

$\boxed{\text{So electron is 1.36 times more likely to be at } x=0 \text{ than to be at one of the classical turning points.}}$

Problem 3

Tunneling probability is

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(V-E)}\,\cdot\,\Delta x} \equiv e^{-2S}$$

$T_{right} > T_{left} \Rightarrow S_{right} < S_{left} \Rightarrow$

$$\sqrt{(2V_0-E)}\cdot a < \sqrt{(V_0-E)}\cdot 2a \Rightarrow$$

$\Rightarrow (2V_0-E)\,a^2 < (V_0-E)\,4a^2 \Rightarrow 2V_0 - E < 4V_0 - 4E \Rightarrow$

$\Rightarrow 3E < 2V_0 \Rightarrow \boxed{E < \frac{2}{3}V_0}$ (a)

So for $E < \frac{2}{3}V_0$ it is more likely to escape through the right barrier
" $E > \frac{2}{3}V_0$ " " " left "

(b) $\frac{\hbar^2\pi^2}{2mL^2}n^2 = E_n = \frac{2}{3}V_0 \Rightarrow n^2 = \frac{2}{3}V_0 \cdot \frac{1}{\pi^2}\cdot\frac{L^2}{3.81\,eV\,\mathring{A}^2} =$

$$= \frac{2}{3}\times 127\,\cancel{eV}\times\frac{1}{\pi^2}\,\frac{\times 100\,\cancel{\mathring{A}^2}}{3.81\,\cancel{eV}\,\cancel{\mathring{A}^2}} = 225$$

$\Rightarrow \boxed{n = 15}$

(c) For a finite square well, energy levels are <u>lower</u>. So electron in level $n = 15$ is more likely to escape through the right rather than the left barrier since its $E < \frac{2}{3}V_0$.