The $a_x - t$ Graph For Constant $a_x$

\[ a_x = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \text{constant} \]

Start clock at $t_1 = 0$, observe again at time $t_2 = t$

Call $v_{0x}$ the velocity at $t_1 = 0$, $v_x$ the velocity at $t_2 = t$

\[ a_x = \frac{v_x - v_{0x}}{t - 0} \]

\[ \Rightarrow \quad v_x = v_{0x} + a_x t \quad ; \quad a_x = \text{constant} \]
The $v_x$-$t$ Graph For Constant $a_x$

$$v_x = v_{0x} + a_x t$$ \quad ; \quad a_x = \text{constant}$$
Evolution of $x$ vs $t$ when $a_x =$Constant

At time $t_1 = 0$, object at $x = x_0$, has $v_{x1} = v_{0x}$

At time $t_2 = t$, object at $x = x$, has $v_{x2} = v_x$

then $v_{av-x} = \frac{x - x_0}{t}$

When $a_x =$ constant, velocity changes at const rate

so for time interval $0 \leq t$, $v_{av-x} = \frac{v_{0x} + v_x}{2}$

But since $v_x = v_{0x} + a_x t \Rightarrow v_{av-x} = \frac{1}{2} (v_{0x} + v_{0x} + a_x t)$

$= v_{0x} + \frac{1}{2} a_x t$
Evolution of $x$ vs $t$ when $a_x$ = Constant

$V_{av-x} = \frac{x - x_0}{t}$

$= v_{0x} + \frac{1}{2}a_xt$

$\Rightarrow x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$

$\Rightarrow$ Parabolic curve in $x$-$t$
Relating $x$, $v_x$ & $a_x$ (without time $t$)

write \[ t = \frac{v_x - v_{0x}}{a_x} \]

substitute in \[ x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \]

\[ \Rightarrow x = x_0 + v_{0x} \left( \frac{v_x - v_{0x}}{a_x} \right) + \frac{1}{2}a_x \left( \frac{v_x - v_{0x}}{a_x} \right)^2 \]

\[ \Rightarrow (x - x_0)2a_x = 2v_{0x}v_x - 2v_{0x}^2 + v_x^2 - 2v_{0x}v_x + v_{0x}^2 \]

\[ \Rightarrow v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \]
An Expression Without $a_x$

Since $v_{av-x} = \frac{x - x_0}{t}$ and $v_{av-x} = \frac{v_{0x} + v_x}{2}$

$$\Rightarrow \frac{x - x_0}{t} = \frac{v_{0x} + v_x}{2}$$

$$\Rightarrow x - x_0 = \left(\frac{v_{0x} + v_x}{2}\right)t$$

This is a useful expression to have when $a_x = \text{constant but unknown}$
Equations For $a=\text{constant}$

\[
\begin{align*}
\vec{v}_x &= \vec{v}_{0x} + a_x t \\
x &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \\
v_x^2 &= v_{0x}^2 + 2a_x (x - x_0) \\
x - x_0 &= \left( \frac{v_{0x} + v_x}{2} \right) t
\end{align*}
\]
Motorcyclist going east, accelerates after passing signpost.
He accelerates at 4.0 m/s². At t=0, he is 5.0 m east of signpost, moving east 15 m/s. (a) find his position and velocity at t=2.0 s.
Where is motorcyclist when his velocity is 25 m/s?

Take signpost as origin of coordinate (x=0), East ® +x

At t=0, \(x_0=5.0\text{ m}, v_{0x}=15\text{ m/s}; a_x=4.0\text{ m/s}^2\)

(a) what is \(x, v_x\) at t=2.0 s, (b) \(x\) when \(v_x=25\text{ m/s}\)

(a) Use \(x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2\)

\[
= 5.0\text{ m} + (15\text{ m/s})(2.0\text{ s}) + \frac{1}{2} (4.0\text{ m/s}^2)(2.0\text{ s})^2 = 43\text{ m}
\]

Velocity at \(x=43\text{ m}: \quad v_x = v_{0x} + a_x t = 23\text{ m/s}\)

(b) no t given! so use \(v_x^2 = v_{0x}^2 + 2a_x (x-x_0)\)

\[
\Rightarrow x = x_0 + \frac{v_x^2 - v_{0x}^2}{2a_x} = 5.0\text{ m} + \frac{(25\text{ m/s})^2 - (15\text{ m/s})^2}{2a_x} = 55\text{ m}
\]
Motion With Constant Acceleration: Freely Falling Bodies

Aristote (4 BC) believed (didn’t check!) that heavier objects fall faster through a medium than lighter ones.

19 centuries later, Galileo did some experiments, disproved this by asserting that all objects falling freely experience a downward acceleration that is constant and independent of object’s weight.
Galileo’s Famous Experiments

Leaning Tower of Pisa

Motion of a Ball on an Inclined plane
Free Fall From Pisa Tower

• Examine a falling object
• Free fall: An idealization of the motion where one ignores “small” effects like
  – Air
  – Earth’s rotation
  – Altitude at location etc
• Free fall is motion with constant acceleration
  – Down or up
• Acceleration $g = -9.8 \text{ m/s}^2$ on earth, $-1.6\text{ m/s}^2$ on moon & $-270 \text{ m/s}^2$ on the sun
Free Fall: Galileo Chose Well

Air-filled tube

Evacuated tube
Inclined Plane Demo By Galileo

Skeptics search Aristotle’s Writing for rebuttal

Galileo Prof. of Pisa

Don Giovanni’s mom with Tuscan noblemen

Assistant using his pulse as a clock

Giuseppe Bezzuoli, Tribuna di Galileo, Firenze
You throw a ball vertically upwards from roof of a building.
Ball leaves your hand at point even with the roof railing with
an upward speed of 15.0m/s; ball is then in free fall. On its way back down it
just misses the railing. Acceleration due to gravity \( g = -9.80 \text{m/s}^2 \). Find the
position and velocity of ball 1.00s and 4.00s after leaving your hand?

Motion is in straight line
but vertical (y axis).
y=0 is at the roof and
+y direction is upwards

Initial position \( y_0 = 0 \),
\( v_{0y} = +15.0 \text{m/s} \),
\( a_y = g = -9.80 \text{m/s}^2 \) down

Find \( x \) & \( v \) at \( t=1.00s,4.00s \)
Position $y$ and velocity $v_y$ after ball leaves hand obtained

from $y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$; $v_y = v_{0y} + a_yt$ \[a_y = g = -9.80 \text{m/s}^2\]

$y(t = 1s) = 0 + (15.0 \text{m/s})(1s) + \frac{1}{2}(-9.80 \text{m/s}^2)(1s)^2$

$= +10.1 \text{m (above roof)}$

$v_y(t = 1s) = 15.0 \text{m/s} + (-9.80 \text{m/s}^2)(1s)$

$= +5.2 \text{m/s (going upwards)}$

Similarly:

$y(t = 4s) = -18.4 \text{m (below roof)}$

$v_y(t = 4s) = -24.2 \text{m/s (going downwards)}$

This is due to the pull of gravity!
Description With y-t and v-t Graphs
Now: Case when $a = a(t) \neq \text{constant}$

Use Calculus, divide interval between $t_1$ & $t_2$ in slices of $\Delta t$

Change in velocity $\Delta v_x = a_{av-x} \Delta t$

= area of shaded strip with height $a_{av-x}$ & width $\Delta t$

Total velocity change from $t_1 \circ t_2$

= total area under $a_x - t$ curve between vertical lines $t_1$ & $t_2$

In Calculus parlance, as $\Delta t \to 0$:

$$v_{2x} - v_{1x} = \int_{v_{1x}}^{v_{2x}} dv_x = \int_{t_1}^{t_2} a_x dt$$

Velocity change = integral of $a_x$ with $t$
Case when $a = a(t) \neq \text{constant}$

Similarly since $v_x = \frac{dx}{dt} \Rightarrow dx = v_x dt$

The change in position $x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v_x dt$

In Conclusion:

$v_x = v_{0x} + \int_{t=0}^{t=t} a_x dt$ and $x = x_0 + \int_{t=0}^{t=t} v_x dt$
Sally is moving at 10 m/s in the x direction when she passes signpost at x = 50 m. Her acceleration is $a = a(t) = 2.0 \text{m/s}^2 - (0.10 \text{m/s}^3) t$.

Find the expression for $v$ and $x$ vs $t$.)

(a) What is the maximum speed she can achieve, and how much time does it take to reach this maximum speed?

At $t = 0$, $x_0 = 50 \text{m}$, $v_{0x} = 10 \text{m/s}$, find $v_x = v_x(t)$

$$v_x = 10 \text{m/s} + \int [2.0 \text{m/s}^2 - (0.10 \text{m/s}^3) t] \, dt$$

Use $\int t^n \, dt = \frac{t^{n+1}}{n+1}$ to get:

$$v_x = 10 \text{m/s} + (2.0 \text{m/s}^2)t - \frac{1}{2}(0.10 \text{m/s}^3)t^2$$

And $x = 50 + \int [10 \text{m/s} + (2.0 \text{m/s}^2)t - \frac{1}{2}(0.10 \text{m/s}^3)t^2] \, dt$

$$x = 50 + (10 \text{m/s})t + \frac{1}{2}(2.0 \text{m/s}^2)t^2 - \frac{1}{2 \times 3}(0.10 \text{m/s}^3)t^3$$

Maximum value of $v_x$ when $\frac{dv_x}{dt} = a_x = 0$, Using $v_x$ expression

$$a_x = 0 = 2.0 \text{m/s}^2 - (0.10 \text{m/s}^3)t \implies t = 20 \text{s}$$
Graph of $x, v_x$ & $a_x$

$$v_{x\text{ max}} = v_x(t = 20s) = 10\text{m/s} + (2.0\text{m/s}^2)(20s) + \frac{1}{2}(0.10\text{m/s}^3)(20s)^2$$

$$= 30\text{m/s}$$

To get position $x$ at $t=20s$ when $v_x=$maximum

Input $t = 20s$ in $x(t) = 50\text{m} + (10\text{m/s})t + \frac{1}{2}(2.0\text{m/s}^2)t^2 - \frac{1}{6}(0.10\text{m/s}^3)t^3$

$$= 517\text{m}$$
A lunar lander is making its descent to moon base. The lander descends slowly under the retro-thrust of its descent engine. The engine is cut off when the lander is 5.0m above surface and has a downward speed of 0.8m/s. With the engine off, the lander is in free fall. What is the speed of the lander just before it touches surface. $g_{moon} = 1.6\text{m/s}^2$

Apply constant acceleration equations to the motion of the lander

Let downward be positive. Lander is in freefall $\Rightarrow a_y = g_{moon}$

What we know: $v_{0y} = +0.8\text{m/s}$, $y-y_0 = 5.0\text{m}$, $a_y = 1.6\text{m/s}^2$, no idea about t!

Use $v_y^2 = v_{0y}^2 + 2a_y(y-y_0)$

$\Rightarrow v_y = \sqrt{v_{0y}^2 + 2a_y(y-y_0)} = \sqrt{(0.8\text{m/s})^2 + 2(1.6\text{m/s}^2)(5.0\text{m})}$

$= 4.1\text{m/s}$

The same descent on Earth would have led to $v_y = 9.9\text{m/s}$ due to the stronger acceleration due to gravity $g$. 
Coming to a lecture near you!
Spiderman steps from the top of a tall building. He falls freely from rest to the ground a distance of \( h \). He falls a distance of \( h/4 \) in the last 1.0s of his fall. What is the height \( h \) of the building?

**Which equation to use? depends on what we know?**

Divide Spidey's motion in 2 segments: \( y = 0 \)  \( y = 3/4h \) and \( y = 3/4h \)  \( h \)

Motion from roof to \( h/4 \) above ground  \( \Rightarrow \)  \( y - y_0 = 3/4h, v_0 = 0, a_y = g \)

Use  \( v_y^2 = v_{y0}^2 + 2a_y(y - y_0) \)

So we get: \( v_y^2 = 0 + a_y(3/4h) \Rightarrow v_y = \sqrt{2a_y(3/4h)} = 3.834\sqrt{h}/\sqrt{m/s} \)

Spiderman's speed after he has fallen for \( 3/4h \) is  \( v_y = 3.83\sqrt{h}/\sqrt{m/s} \)

In the next segment, \( y - y_0 = h/4, v_{y0} = 3.83\sqrt{h}/\sqrt{m/s}, a_y = g, t=1s \)

Clearly we should use: \( y = y_0 + v_{y0}t + (1/2)a_yt^2 \)

\[ \Rightarrow (h/4) = 3.83\sqrt{h}/\sqrt{m} + 4.90m. \]

Now solve for \( h \) but how?

Write as  \( \frac{1}{4}u^2 - 3.83u\sqrt{m} - 4.90m = 0 \), solve for \( u \) (quadratic eq)

\[ if \ au^2 + bu + c = 0 \Rightarrow u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Taking the positive root  \( \Rightarrow u = 16.52\sqrt{m} \Rightarrow h = u^2 = 273m \)
A Groovy Crash!
Helicopter carrying Dr. Evil takes off with a constant upward acceleration of 5.0m/s^2. Austin Powers jumps on just as the helicopter lifts off the ground. After the two men struggle for 10.0s, Powers shuts off the engine and steps out of the chopper. Assume that the chopper is in free fall after its engine is shut off, and ignore the effects of air-resistance. (a) what is the max. height above ground reached by chopper. (b) Powers deploys a jet pack strapped on his back 7.0s after leaving chopper, and then has a constant downward acceleration with magnitude 2.0m/s^2. How far is Powers above the ground when the chopper crashes to the ground?

- Analyze the action segments in the narrative
  - Chopper/Dr.Evil
    - 10.0s under constant upward acceleration of 5m/s^2
    - Followed by free-fall under gravity
      - They continue to go up and then come down
  - Austin Powers
    - 10.0s under constant upward acceleration
    - Followed by free-fall under gravity for 7.0s,
    - Followed by constant downward acceleration 2m/s^2
- Set up coordinate system, ground = y =0, up is +y
what is the max. height above ground reached by chopper?
When engine shuts off, both chopper + Austin + Dr. Evil have:
Same upward velocity \( v_y = v_{oy} + a_y t = 0 + (5 \text{m/s}^2)(10 \text{s}) = 50 \text{m/s} \)
Same height \( y = y_o + v_{oy} t + \frac{1}{2}a_y t^2 = 0 + 0 + 0.5(5.0 \text{ m/s}^2)(10 \text{s})^2 = 250 \text{m} \)

How much more do the chopper/Dr.Evil & Austin Powers climb?
When engine shuts off: chopper etc continue to climb against gravity, till \( v_y = 0 \)
\[ \Rightarrow v_{oy} = +50 \text{m/s}, \ y_o = 250 \text{m}, a_y = -9.8 \text{m/s}^2 \text{and when} \ v_y = 0 \text{m/s, what is} \ y? \]

Use \( v_y^2 = v_{oy}^2 + 2a_y(y-y_o) \Rightarrow 0 = v_{oy}^2 + 2a_y(y-y_o) \Rightarrow y = \frac{v_y^2 - v_{oy}^2}{2a_y} + y_o = \frac{0 - (50 \text{m/s})^2}{2(-9.8 \text{m/s}^2)} + 250 \text{m} \)
\[ \Rightarrow y = \text{max height attained} = 378 \text{m}. \text{From this height every thing starts to fall down} \]

Time \( t \) for chopper to crash to \( y=0 \) under free-fall from height of \( y_o = +250 \text{m} \) and \( v_{oy} = +50 \text{m/s}? \)
Use \( (y-y_o) = v_{oy} t + 0.5a_y t^2 \Rightarrow -250 = (50 \text{m/s})t - 0.5(9.8 \text{m/s}^2)t^2 \) ...solve this quadratic eqn for \( t \)

+ solution of \( 0.5(9.8 \text{m/s}^2)t^2 - (50 \text{m/s})t - 250 = 0 \) is \( t = (1/9.8)(50 + \sqrt{(50)^2 - 4(0.5)(-250)}) \text{s} \)
Time is always positive so \( t = 13.9 \text{s} \) after Powers shuts down Dr. Evil's engine!
Now let's look at Mr. Powers' trajectory: when he steps out of the chopper, he retains the initial velocity of the chopper. But his acceleration changes from $+5\text{m/s}^2$ to $-9.8\text{m/s}^2$. Without the jet pack he would have crashed at the same time as the chopper. Dr. Evil. But when he turns on the jetpack after 7s of free fall, his acceleration changes to $-2\text{m/s}^2$ instead of $-9.8\text{m/s}^2$ \Rightarrow he descends to ground will smaller velocity and travels less vertically \Rightarrow he is still above ground when the chopper crashes. How high above ground is he when the chopper crashes? \Rightarrow solution in 2 steps!

After 7s of free fall, he is at $y = y_0 + v_{0y}t + 0.5a_yt^2 = (250\text{m}) + (50\text{m/s})(7\text{s}) - (0.5)(9.8\text{m/s}^2)(7\text{s})$
\Rightarrow $y-y_0 = 360\text{m} \Rightarrow 360\text{m}$ above ground, but he is not out of danger yet!
His velocity at that height is $v_y = v_{0y} + a_yt = 50\text{m/s} + (-9.8\text{m/s}^2)(7\text{s}) = -18.6\text{m/s}$

$(13.9-7.0=6.9)$s before Dr. Evil crashes, Austin fires jet pack, forcing him to accelerate at $a_y = -2\text{m/s}^2$.
After that 6.9s, he is at $y = y_0 + v_{0y}t + 0.5a_yt^2 = 360\text{m} + (-18.6\text{m/s})(6.9\text{s}) + 0.5(-2\text{m/s}^2)(6.9\text{m/s}^2) = 184\text{m}$
\Rightarrow Austin Powers is safe at a height above ground of 184m when the chopper with Dr. Evil crashes!
Relative Velocity makes mid-air refueling possible!
Relative Velocity

Blue Angel pilots must keep track of their velocity w.r.t air so as to maintain enough airflow over their wings to sustain the “lift” & not crash.

They must also be aware of relative velocity of their aircraft w.r.t another!
Frames of Reference, Observers & Motion

Event: Some thing happening, some where at some time

Frame of reference $S = \text{a coordinate system + clock}$
Observer $O$ : sits in $S$, measures events with ruler, clock
Observers in diff frames of refs, depending on relative location may measure different positions for an event but measure same time. Their clocks are synchronized!

Observers can move w.r.t each other
Relative Velocity in 1 Dimension

Woman walks with velocity of 1.0 m/s along train's aisle
Train is moving with velocity of 3.0 m/s to the right

What is the woman's velocity? According to which Observer???
Relative Velocity in 1 Dimension

Event: Woman
Two observers
- cyclist (A)
- on train (B)

At any instant (take a snapshot)

\[ x_{P/A} = \text{pos. of } P \text{ rel. to frame } A; \quad x_{P/B} = \text{pos. of } P \text{ rel. to frame } B \text{ (train)} \]

\[ x_{B/A} = \text{distance from origin of } A \text{ to origin of } B \]

Clearly \[ x_{P/A} = x_{P/B} + x_{B/A} \] and \[ \frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \]

So woman's velocity as seen by cyclist (A)

\[ v_{P/A} = 1.0 \text{m/s} + 3.0 \text{m/s} = 4.0 \text{m/s} \]

but If woman was walking in opp. dir. in train

\[ v_{P/A} = -1.0 \text{m/s} + 3.0 \text{m/s} = 2.0 \text{m/s} \]