Physics 2a, Nov 17, lecture 25

⋆Reading: chapters 9 and 10.

• Last time, Examples of moments of inertia:

$$I = \begin{cases} \frac{1}{12}ML^2 & \text{rod through center} \\ \frac{1}{3}ML^2 & \text{rod through end} \\ \frac{1}{2}MR^2 & \text{solid cylinder} \\ \frac{2}{5}MR^2 & \text{solid sphere} \\ \frac{2}{3}MR^2 & \text{thin walled hollow sphere.} \end{cases}$$

For cylinder or sphere of radius R, write $I = cMR^2$, and note that $c_{hollow} > c_{solid}$ and $c_{cylinder} > c_{sphere}$ make intuitive sense, since bigger c means more mass farther from the axis of rotation. (Parallel axis result: the moment around an axis parallel to, and at a distance d from, one going through the CM is $I_p = I_{cm} + Md^2$. For example, the I of a rod through an end vs through the center are related this way.)

Race round rigid bodies down an incline plane, which wins? Use conservation of energy. $E_{initial} = Mgh$. $E_{final} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$, and $\omega = v_{cm}/R$ (rolling without slipping), so $E_{final} = \frac{1}{2}(1+c)Mv_{cm}^2$, so $v_{cm} = \sqrt{2gh/(1+c)}$. Smaller I object wins. Makes sense, less energy taken up with rotation means more going into velocity. Writing $h = d\sin\beta$ where d is the distance traveled along the slope shows that the acceleration along the slope is $a = g\sin\beta/(1+c)$.

Unwinding cable example. Mass m on string, wrapped around cylinder with mass M and radius R. Mass drops height h. Find it's speed.

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I(v/R)^2$, so $v = \sqrt{2gh/(1 + I/mR^2)}$, with $I = \frac{1}{2}MR^2$. Note that $v^2 = 2ah$, with $a = g/(1 + I/mR^2)$.

• Let's now reconsider the above examples, as illustrations of the use of torque, $\tau = \vec{r} \times \vec{F}$.

Consider first the unwinding cable example. The downward force on the mass is $mg - T = ma$. The tension T provides a torque $\tau = TR = I\alpha$ on the cylinder. Finally, $a = R\alpha$. Solve these to get $a = g/(1 + I/mR^2)$.

Now consider the rolling body example. The force parallel to the slope is $Mg\sin\beta - f_f = Ma$. The torque around the middle is $\tau = Rf_f = I\alpha$. Setting $a = \alpha R$ for non-slipping, get $g\sin\beta = a(1+c)$, where $c = I/MR^2$, and this agrees with the acceleration found last time using energy considerations. Note that $f_f = cMg\sin\beta/(1+c)$ and $n = Mg\cos\beta$, so need minimum friction coefficient $\mu_s = \frac{c}{1+c}\tan\beta$.

• More on angular momentum, $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{L}_{cm} + I\vec{\omega}$, and examples. Note that it depends on choice of origin. As seen in Monday's lecture, $\vec{\tau} = \frac{d\vec{L}}{dt}$.

If no torque, $\vec{\tau} = 0$, angular momentum is conserved, $\vec{L} =$ constant.

Two objects, A and B, since $\vec{F}_{A\to B} = -\vec{F}_{B\to A}$, we see $\vec{\tau}_{A\to B} = -\vec{\tau}_{B\to A}$, equal and opposite torques, so $\frac{d}{dt}(\vec{L}_A + \vec{L}_B) = 0$. In general, Newton's 3rd law $\to \vec{\tau}_{total} = \vec{\tau}_{external}$, which vanishes for a closed system. So closed systems have conserved angular momentum. At a fundamental level, angular momentum is always conserved, though it can flow in and out of a system. Conservation of angular momentum is a deep principle, like conservation of energy and conservation of momentum. (They are related to symmetries: energy to time translations, momentum to space translations, and angular momentum to rotational invariance).

• Spinning with dumbbells, bring them in and use conservation of L to find ω_f. Compare $K_f - K_i$ to work done.

• Bullet in door example. Door width d and mass M. Bullet of mass m and velocity v hits at distance ℓ from hinge. Using conservation of $\vec{L}$, get $L_z = mv\ell$ before, and $\vec{L} = I\omega$ after, where $I = \frac{1}{3}Md^2 + m\ell^2$. Equating gives $\omega = mv\ell/I$. Note $K_{before} = \frac{1}{2}mv^2$ and $K_{after} = \frac{1}{2}I\omega^2$, and $K_{before} - K_{after}$ is positive, as expected, and equal to the energy lost to heat in the inelastic collision of bullet and door.

• Gyroscopes and precession. The weight of the gyro leads to $\vec{\tau} = \vec{r} \times \vec{w}$. This is perpendicular to $\vec{L}$ (since $\vec{L}$ is parallel to $\vec{r}$), so $\frac{d}{dt}(\vec{L}\cdot\vec{L}) = 0$, the magnitude of $\vec{L}$ is unchanged, but it's direction rotates in a circle. The procession angular speed is $\Omega = |\vec{d|}/|\vec{L}|/dt = Mgr/I\omega$.