Physics 2a, Nov 17, lecture 25

⭐Reading: chapters 9 and 10.

• Last time, Examples of moments of inertia:

\[
I = \begin{cases} 
\frac{1}{12}ML^2 & \text{rod through center} \\
\frac{1}{3}ML^2 & \text{rod through end} \\
\frac{1}{3}MR^2 & \text{solid cylinder} \\
\frac{2}{5}MR^2 & \text{solid sphere} \\
\frac{2}{3}MR^2 & \text{thin walled hollow sphere}.
\end{cases}
\]

For cylinder or sphere of radius \(R\), write \(I = cMR^2\), and note that \(c_{\text{hollow}} > c_{\text{solid}}\) and \(c_{\text{cylinder}} > c_{\text{sphere}}\) make intuitive sense, since bigger \(c\) means more mass farther from the axis of rotation. (Parallel axis result: the moment around an axis parallel to, and at a distance \(d\) from, one going through the CM is \(I_p = I_{cm} + Md^2\). For example, the \(I\) of a rod through an end vs through the center are related this way.)

Race round rigid bodies down an incline plane, which wins? Use conservation of energy. \(E_{initial} = Mgh\). \(E_{final} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2\), and \(\omega = v_{cm}/R\) (rolling without slipping), so \(E_{final} = \frac{1}{2}(1+c)Mv_{cm}^2\), so \(v_{cm} = \sqrt{2gh/(1+c)}\). Smaller \(I\) object wins.

Makes sense, less energy taken up with rotation means more going into velocity. Writing \(h = d\sin\beta\) where \(d\) is the distance traveled along the slope shows that the acceleration along the slope is \(a = g\sin\beta/(1+c)\).

Unwinding cable example. Mass \(m\) on string, wrapped around cylinder with mass \(M\) and radius \(R\). Mass drops height \(h\). Find it’s speed.

\(mgh = \frac{1}{2}mv^2 + \frac{1}{2}I(v/R)^2\), so \(v = \sqrt{2gh/(1 + I/mR^2)}\), with \(I = \frac{1}{2}MR^2\). Note that \(v^2 = 2ah\), with \(a = g/(1 + I/mR^2)\).

• Let’s now reconsider the above examples, as illustrations of the use of torque, \(\tau = \vec{r} \times \vec{F}\).

Consider first the unwinding cable example. The downward force on the mass is \(mg - T = ma\). The tension \(T\) provides a torque \(\tau = TR = Ia\) on the cylinder. Finally, \(a = R\alpha\). Solve these to get \(a = g/(1 + I/mR^2)\).

Now consider the rolling body example. The force parallel to the slope is \(Mg\sin\beta - f_f = Ma\). The torque around the middle is \(\tau = Rf_f = I\alpha\). Setting \(a = \alpha R\) for non-slipping, get \(g\sin\beta = a(1+c)\), where \(c = I/MR^2\), and this agrees with the acceleration found last time using energy considerations. Note that \(f_f = cMg\sin\beta/(1+c)\) and \(n = Mg\cos\beta\), so need minimum friction coefficient \(\mu_s = \frac{c}{1+c}\tan\beta\).
• More on angular momentum, \( \vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{L}_{cm} + I \vec{\omega} \), and examples. Note that it depends on choice of origin. As seen in Monday’s lecture, \( \vec{\tau} = \frac{d\vec{L}}{dt} \).

If no torque, \( \vec{\tau} = 0 \), angular momentum is conserved, \( \vec{L} = \text{constant} \).

Two objects, \( A \) and \( B \), since \( \vec{F}_{A \rightarrow B} = -\vec{F}_{B \rightarrow A} \), we see \( \vec{\tau}_{A \rightarrow B} = -\vec{\tau}_{B \rightarrow A} \), equal and opposite torques, so \( \frac{d}{dt}(\vec{L}_A + \vec{L}_B) = 0 \). In general, Newton’s 3rd law \( \rightarrow \vec{\tau}_{\text{total}} = \vec{\tau}_{\text{external}} \), which vanishes for a closed system. So closed systems have conserved angular momentum. At a fundamental level, angular momentum is always conserved, though it can flow in and out of a system. Conservation of angular momentum is a deep principle, like conservation of energy and conservation of momentum. (They are related to symmetries: energy to time translations, momentum to space translations, and angular momentum to rotational invariance).

• Spinning with dumbbells, bring them in and use conservation of \( L \) to find \( \omega_f \). Compare \( K_f - K_i \) to work done.

• Bullet in door example. Door width \( d \) and mass \( M \). Bullet of mass \( m \) and velocity \( v \) hits at distance \( \ell \) from hinge. Using conservation of \( \vec{L} \), get \( L_z = mv\ell \) before, and \( \vec{L} = I \vec{\omega} \) after, where \( I = \frac{1}{3}Md^2 + ml^2 \). Equating gives \( \omega = \frac{mv\ell}{I} \). Note \( K_{\text{before}} = \frac{1}{2}mv^2 \) and \( K_{\text{after}} = \frac{1}{2}I\omega^2 \), and \( K_{\text{before}} - K_{\text{after}} \) is positive, as expected, and equal to the energy lost to heat in the inelastic collision of bullet and door.

• Gyroscopes and precession. The weight of the gyro leads to \( \vec{\tau} = \vec{r} \times \vec{w} \). This is perpendicular to \( \vec{L} \) (since \( \vec{L} \) is parallel to \( \vec{r} \)), so \( \frac{d}{dt}(\vec{L} \cdot \vec{L}) = 0 \), the magnitude of \( \vec{L} \) is unchanged, but it’s direction rotates in a circle. The precession angular speed is \( \Omega = \frac{|\vec{d}|}{|\vec{L}|/dt = Mgr/I\omega} \).