Chapter 17

Physics 110A-B Exams

The following pages contain problems and solutions from midterm and final exams in Physics 110A-B.

17.1 F05 Physics 110A Midterm #1

[1] A particle of mass m moves in the one-dimensional potential

$$U(x) = U_0 \, \frac{x^2}{a^2} \, e^{-x/a} \; . \tag{17.1}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.

(b) Sketch a representative set of phase curves. Identify and classify any and all fixed points. Find the energy of each and every separatrix.

(c) Sketch all the phase curves for motions with total energy $E = \frac{2}{5}\,U_0$. Do the same for $E = U_0$. (Recall that $e = 2.71828\ldots$.)

(d) Derive and expression for the period T of the motion when $|x| \ll a$.

Solution:

(a) Clearly $U(x)$ diverges to $+\infty$ for $x \to -\infty$, and $U(x) \to 0$ for $x \to +\infty$. Setting $U'(x) = 0$, we obtain the equation

$$U'(x) = \frac{U_0}{a^2} \left(2x - \frac{x^2}{a} \right) e^{-x/a} = 0 \; , \tag{17.2}$$

with (finite x) solutions at $x = 0$ and $x = 2a$. Clearly $x = 0$ is a local minimum and $x = 2a$ a local maximum. Note $U(0) = 0$ and $U(2a) = 4\,e^{-2}\,U_0 \approx 0.541\,U_0$.

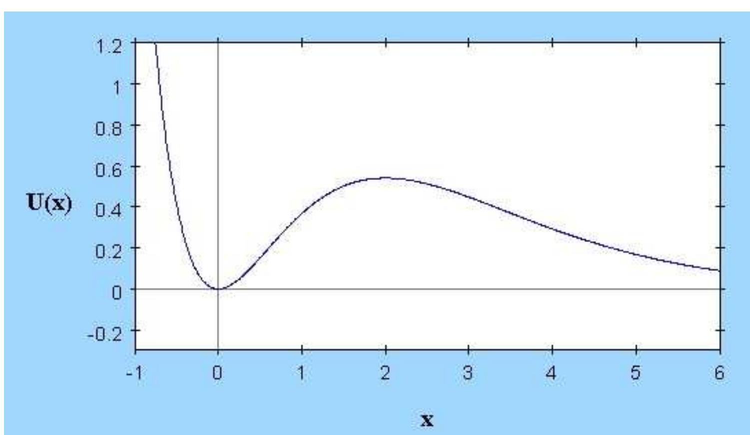

Figure 17.1: The potential $U(x)$. Distances are here measured in units of a, and the potential in units of U_0.

(b) Local minima of a potential $U(x)$ give rise to centers in the (x, v) plane, while local maxima give rise to saddles. In Fig. 17.2 we sketch the phase curves. There is a center at

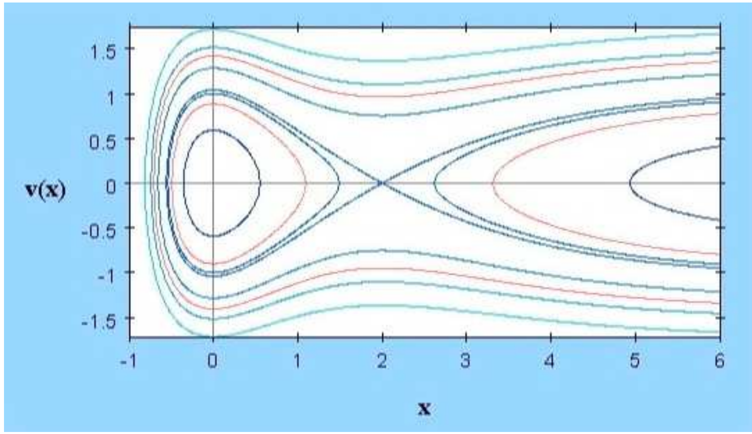

Figure 17.2: Phase curves for the potential $U(x)$. The red curves show phase curves for $E = \frac{2}{5}\,U_0$ (interior, disconnected red curves, $|v| < 1$) and $E = U_0$ (outlying red curve). The separatrix is the dark blue curve which forms a saddle at $(x, v) = (2, 0)$, and corresponds to an energy $E = 4\,e^{-2}\,U_0$.

$(0, 0)$ and a saddle at $(2a, 0)$. There is one separatrix, at energy $E = U(2a) = 4\,e^{-2}\,U_0 \approx 0.541\,U_0$.

(c) Even without a calculator, it is easy to verify that $4\,e^{-2} > \frac{2}{5}$. One simple way is to multiply both sides by $\frac{5}{2}\,e^2$ to obtain $10 > e^2$, which is true since $e^2 < (2.71828\ldots)^2 < 10$. Thus, the energy $E = \frac{2}{5}\,U_0$ lies below the local maximum value of $U(2a)$, which means that there are *two* phase curves with $E = \frac{2}{5}\,U_0$.

It is also quite obvious that the second energy value given, $E = U_0$, lies above $U(2a)$, which means that there is a single phase curve for this energy. One finds bound motions only for $x < 2$ and $0 \le E < U(2a)$. The phase curves corresponding to total energy $E = \frac{2}{5}\,U_0$ and $E = U_0$ are shown in Fig. 17.2.

(d) Expanding $U(x)$ in a Taylor series about $x = 0$, we have

$$U(x) = \frac{U_0}{a^2}\left\{x^2 - \frac{x^3}{a} + \frac{x^4}{2a^2} + \ldots\right\} . \tag{17.3}$$

The leading order term is sufficient for $|x| \ll a$. The potential energy is then equivalent to that of a spring, with spring constant $k = 2U_0/a^2$. The period is

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ma^2}{2U_0}}} . \tag{17.4}$$

[2] A forced, damped oscillator obeys the equation

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f_0\,\cos(\omega_0 t)\ . \tag{17.5}$$

You may assume the oscillator is underdamped.

(a) Write down the most general solution of this differential equation.

(b) Your solution should involve two constants. Derive two equations relating these constants to the initial position $x(0)$ and the initial velocity $\dot{x}(0)$. *You do not have to solve these equations.*

(c) Suppose $\omega_0 = 5.0\,\mathrm{s}^{-1}$, $\beta = 4.0\,\mathrm{s}^{-1}$, and $f_0 = 8\,\mathrm{cm\,s}^{-2}$. Suppose further you are told that $x(0) = 0$ and $x(T) = 0$, where $T = \frac{\pi}{6}\,\mathrm{s}$. Derive an expression for the initial velocity $\dot{x}(0)$.

Solution: (a) The general solution with forcing $f(t) = f_0\,\cos(\Omega t)$ is

$$x(t) = x_{\rm h}(t) + A(\Omega)\,f_0\,\cos\big(\Omega t - \delta(\Omega)\big)\ , \tag{17.6}$$

with

$$A(\Omega) = \Big[(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\Big]^{-1/2}\quad,\quad \delta(\Omega) = \tan^{-1}\bigg(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\bigg) \tag{17.7}$$

and

$$x_{\rm h}(t) = C\,e^{-\beta t}\,\cos(\nu t) + D\,e^{-\beta t}\,\sin(\nu t)\ , \tag{17.8}$$

with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

In our case, $\Omega = \omega_0$, in which case $A = (2\beta\omega_0)^{-1}$ and $\delta = \frac{1}{2}\pi$. Thus, the most general solution is

$$\boxed{x(t) = C\,e^{-\beta t}\,\cos(\nu t) + D\,e^{-\beta t}\,\sin(\nu t) + \frac{f_0}{2\beta\omega_0}\,\sin(\omega_0 t)}\ . \tag{17.9}$$

(b) We determine the constants C and D by the boundary conditions on $x(0)$ and $\dot{x}(0)$:

$$\boxed{x(0) = C}\quad,\quad \boxed{\dot{x}(0) = -\beta C + \nu D + \frac{f_0}{2\beta}}\ . \tag{17.10}$$

Thus,

$$C = x(0)\quad,\quad D = \frac{\beta}{\nu}\,x(0) + \frac{1}{\nu}\,\dot{x}(0) - \frac{f_0}{2\beta\nu}\ . \tag{17.11}$$

(c) From $x(0) = 0$ we obtain $C = 0$. The constant D is then determined by the condition at time $t = T = \frac{1}{6}\pi$.

Note that $\nu = \sqrt{\omega_0^2 - \beta^2} = 3.0\,\mathrm{s}^{-1}$. Thus, with $T = \frac{1}{6}\pi$, we have $\nu T = \frac{1}{2}\pi$, and

$$x(T) = D\,e^{-\beta T} + \frac{f_0}{2\beta\omega_0}\,\sin(\omega_0 T)\ . \tag{17.12}$$

This determines D:

$$D = -\frac{f_0}{2\beta\omega_0}\, e^{\beta T}\, \sin(\omega_0 T) \ . \tag{17.13}$$

We now can write

$$\begin{aligned} \dot{x}(0) &= \nu D + \frac{f_0}{2\beta} && (17.14) \\ &= \frac{f_0}{2\beta}\left(1 - \frac{\nu}{\omega_0}\, e^{\beta T}\, \sin(\omega_0 T)\right) && (17.15) \\ &= \boxed{\left(1 - \tfrac{3}{10}\, e^{2\pi/3}\right) \text{cm/s}} \ . && (17.16) \end{aligned}$$

Numerically, the value is $\dot{x}(0) \approx 0.145\,\text{cm/s}$.

17.2 F05 Physics 110A Midterm #2

[**1**] Two blocks connected by a spring of spring constant k are free to slide frictionlessly along a horizontal surface, as shown in Fig. 17.3. The unstretched length of the spring is a.

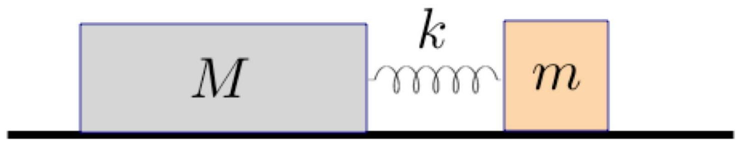

Figure 17.3: Two masses connected by a spring sliding horizontally along a frictionless surface.

(a) Identify a set of generalized coordinates and write the Lagrangian.
`[15 points]`

Solution : As generalized coordinates I choose X and u, where X is the position of the right edge of the block of mass M, and $X + u + a$ is the position of the left edge of the block of mass m, where a is the unstretched length of the spring. Thus, the extension of the spring is u. The Lagrangian is then

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{X}+\dot{u})^2 - \tfrac{1}{2}ku^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}m\dot{u}^2 + m\dot{X}\dot{u} - \tfrac{1}{2}ku^2 \ . \end{aligned} \tag{17.17}$$

(b) Find the equations of motion.
`[15 points]`

Solution : The canonical momenta are

$$p_X \equiv \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{u} \quad , \quad p_u \equiv \frac{\partial L}{\partial \dot{u}} = m(\dot{X}+\dot{u}) \ . \tag{17.18}$$

The corresponding equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad (M+m)\ddot{X} + m\ddot{u} = 0 \tag{17.19}$$

$$\dot{p}_u = F_u = \frac{\partial L}{\partial u} \qquad \Rightarrow \qquad m(\ddot{X}+\ddot{u}) = -ku \ . \tag{17.20}$$

(c) Find all conserved quantities.
`[10 points]`

Solution : There are two conserved quantities. One is p_X itself, as is evident from the fact that L is cyclic in X. This is the conserved 'charge' Λ associated with the continuous symmetry $X \to X + \zeta$. *i.e.* $\Lambda = p_X$. The other conserved quantity is the Hamiltonian H, since L is cyclic in t. Furthermore, because the kinetic energy is homogeneous of degree two in the generalized velocities, we have that $H = E$, with

$$E = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}m\dot{u}^2 + m\dot{X}\dot{u} + \tfrac{1}{2}ku^2 \ . \tag{17.21}$$

It is possible to eliminate $\dot{X}$, using the conservation of Λ:

$$\dot{X} = \frac{\Lambda - m\dot{u}}{M+m} \ . \tag{17.22}$$

This allows us to write

$$E = \frac{\Lambda^2}{2(M+m)} + \frac{Mm\,\dot{u}^2}{2(M+m)} + \tfrac{1}{2}ku^2 \ . \tag{17.23}$$

(d) Find a complete solution to the equations of motion. As there are two degrees of freedom, your solution should involve 4 constants of integration. You need not match initial conditions, and you need not choose the quantities in part (c) to be among the constants. `[10 points]`

Solution : Using conservation of Λ, we may write $\ddot{X}$ in terms of $\ddot{x}$, in which case

$$\frac{Mm}{M+m}\,\ddot{u} = -ku \quad \Rightarrow \quad u(t) = A\cos(\Omega t) + B\sin(\Omega t) \ , \tag{17.24}$$

where

$$\Omega = \sqrt{\frac{(M+m)k}{Mm}} \ . \tag{17.25}$$

For the X motion, we integrate eqn. 17.22 above, obtaining

$$X(t) = X_0 + \frac{\Lambda t}{M+m} - \frac{m}{M+m}\Big(A\cos(\Omega t) - A + B\sin(\Omega t)\Big) \ . \tag{17.26}$$

There are thus four constants: X_0, Λ, A, and B. Note that conservation of energy says

$$E = \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}k(A^2 + B^2) \ . \tag{17.27}$$

Alternate solution : We could choose X as the position of the left block and x as the position of the right block. In this case,

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}k(x - X - b)^2 \ . \tag{17.28}$$

Here, b includes the unstretched length a of the spring, but may also include the size of the blocks if, say, X and x are measured relative to the blocks' midpoints. The canonical momenta are

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \quad , \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ . \tag{17.29}$$

The equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad M\ddot{X} = k(x - X - b) \tag{17.30}$$

$$\dot{p}_x = F_x = \frac{\partial L}{\partial x} \qquad \Rightarrow \qquad m\ddot{x} = -k(x - X - b) \ . \tag{17.31}$$

The one-parameter family which leaves L invariant is $X \to X + \zeta$ and $x \to x + \zeta$, *i.e.* simultaneous and identical displacement of both of the generalized coordinates. Then

$$\Lambda = M\dot{X} + m\dot{x} \ , \tag{17.32}$$

which is simply the x-component of the total momentum. Again, the energy is conserved:

$$E = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}k\,(x - X - b)^2 \ . \tag{17.33}$$

We can combine the equations of motion to yield

$$Mm\,\frac{d^2}{dt^2}\,\big(x - X - b\big) = -k\,(M+m)\big(x - X - b\big) \ , \tag{17.34}$$

which yields

$$x(t) - X(t) = b + A\cos(\Omega t) + B\sin(\Omega t) \ , \tag{17.35}$$

From the conservation of Λ, we have

$$M X(t) + m\,x(t) = \Lambda t + C \ , \tag{17.36}$$

were C is another constant. Thus, we have the motion of the system in terms of four constants: A, B, Λ, and C:

$$X(t) = -\tfrac{m}{M+m}\big(b + A\cos(\Omega t) + B\sin(\Omega t)\big) + \frac{\Lambda t + C}{M+m} \tag{17.37}$$

$$x(t) = \tfrac{M}{M+m}\big(b + A\cos(\Omega t) + B\sin(\Omega t)\big) + \frac{\Lambda t + C}{M+m} \ . \tag{17.38}$$

[2] A uniformly dense ladder of mass m and length 2ℓ leans against a block of mass M, as shown in Fig. 17.4. Choose as generalized coordinates the horizontal position X of the right end of the block, the angle θ the ladder makes with respect to the floor, and the coordinates (x, y) of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder can be written as a sum $T_{\rm CM} + T_{\rm rot}$, where $T_{\rm CM} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\rm rot} = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertial. For a uniformly dense ladder of length 2ℓ, $I = \frac{1}{3}m\ell^2$.

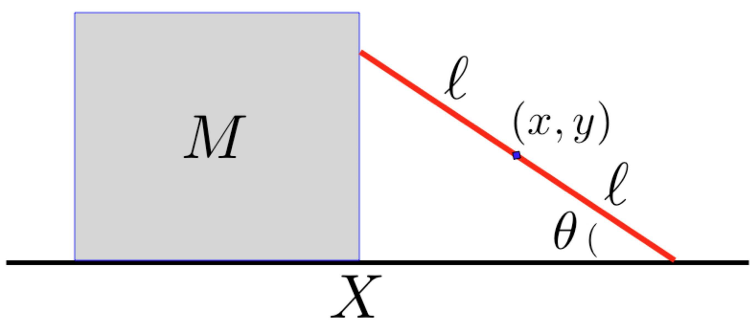

Figure 17.4: A ladder of length 2ℓ leaning against a massive block. All surfaces are frictionless..

(a) Write down the Lagrangian for this system in terms of the coordinates X, θ, x, y, and their time derivatives.
`[10 points]`

Solution : We have $L = T - U$, hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \ . \tag{17.39}$$

(b) Write down all the equations of constraint.
`[10 points]`

Solution : There are two constraints, corresponding to contact between the ladder and the block, and contact between the ladder and the horizontal surface:

$$G_1(X, \theta, x, y) = x - \ell\cos\theta - X = 0 \tag{17.40}$$

$$G_2(X, \theta, x, y) = y - \ell\sin\theta = 0 \ . \tag{17.41}$$

(c) Write down all the equations of motion.
`[10 points]`

Solution : Two Lagrange multipliers, λ_1 and λ_2, are introduced to effect the constraints. We have for each generalized coordinate q_σ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \equiv Q_\sigma \ , \tag{17.42}$$

where there are $k = 2$ constraints. We therefore have

$$M\ddot{X} = -\lambda_1 \tag{17.43}$$

$$m\ddot{x} = +\lambda_1 \tag{17.44}$$

$$m\ddot{y} = -mg + \lambda_2 \tag{17.45}$$

$$I\ddot{\theta} = \ell \sin\theta\, \lambda_1 - \ell\cos\theta\, \lambda_2 \ . \tag{17.46}$$

These four equations of motion are supplemented by the two constraint equations, yielding six equations in the six unknowns $\{X, \theta, x, y, \lambda_1, \lambda_2\}$.

(d) Find all conserved quantities.
`[10 points]`

Solution : The Lagrangian and all the constraints are invariant under the transformation

$$X \to X + \zeta \quad , \quad x \to x + \zeta \quad , \quad y \to y \quad , \quad \theta \to \theta \ . \tag{17.47}$$

The associated conserved 'charge' is

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = M\dot{X} + m\dot{x} \ . \tag{17.48}$$

Using the first constraint to eliminate x in terms of X and θ, we may write this as

$$\Lambda = (M + m)\dot{X} - m\ell \sin\theta\, \dot{\theta} \ . \tag{17.49}$$

The second conserved quantity is the total energy E. This follows because the Lagrangian and all the constraints are independent of t, and because the kinetic energy is homogeneous of degree two in the generalized velocities. Thus,

$$E = \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} I\dot{\theta}^2 + mgy \tag{17.50}$$

$$= \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}\Big(I + m\ell^2 - \tfrac{m}{M+m}\, m\ell^2 \sin^2\theta\Big)\, \dot{\theta}^2 + mg\ell \sin\theta \ , \tag{17.51}$$

where the second line is obtained by using the constraint equations to eliminate x and y in terms of X and θ.

(e) What is the condition that the ladder detaches from the block? You do not have to solve for the angle of detachment! Express the detachment condition in terms of any quantities you find convenient.
`[10 points]`

Solution : The condition for detachment from the block is simply $\lambda_1 = 0$, *i.e.* the normal force vanishes.

Further analysis : It is instructive to work this out in detail (though this level of analysis was not required for the exam). If we eliminate x and y in terms of X and θ, we find

$$x = X + \ell\cos\theta \qquad\qquad y = \ell \sin\theta \tag{17.52}$$

$$\dot{x} = \dot{X} - \ell \sin\theta\, \dot{\theta} \qquad\qquad \dot{y} = \ell\cos\theta\, \dot{\theta} \tag{17.53}$$

$$\ddot{x} = \ddot{X} - \ell \sin\theta\, \ddot{\theta} - \ell\cos\theta\, \dot{\theta}^2 \qquad\qquad \ddot{y} = \ell\cos\theta\, \ddot{\theta} - \ell\sin\theta\, \dot{\theta}^2 \ . \tag{17.54}$$

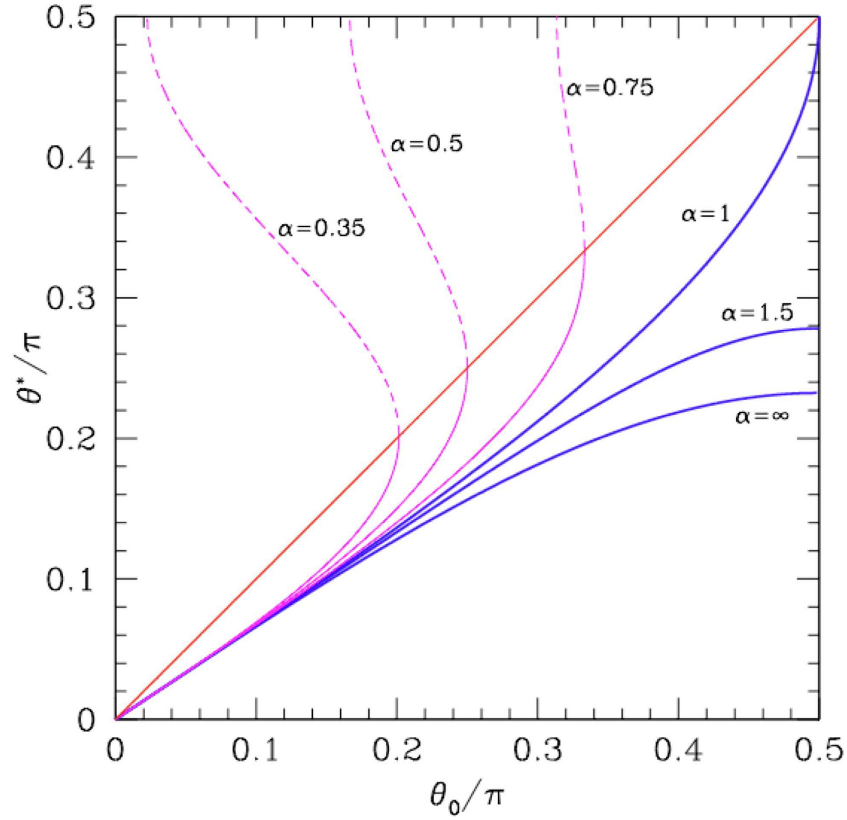

Figure 17.5: Plot of θ^* *versus* θ_0 for the ladder-block problem (eqn. 17.64). Allowed solutions, shown in blue, have $\alpha \geq 1$, and thus $\theta^* \leq \theta_0$. Unphysical solutions, with $\alpha < 1$, are shown in magenta. The line $\theta^* = \theta_0$ is shown in red.

We can now write

$$\lambda_1 = m\ddot{x} = m\ddot{X} - m\ell \sin\theta\,\ddot{\theta} - m\ell\cos\theta\,\dot{\theta}^2 = -M\ddot{X} \ , \tag{17.55}$$

which gives

$$(M+m)\ddot{X} = m\ell\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \ , \tag{17.56}$$

and hence

$$Q_x = \lambda_1 = -\frac{Mm}{m+m}\,\ell\,\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \ . \tag{17.57}$$

We also have

$$\begin{aligned} Q_y = \lambda_2 &= mg + m\ddot{y} \\ &= mg + m\ell\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big) \ . \end{aligned} \tag{17.58}$$

We now need an equation relating $\ddot{\theta}$ and $\dot{\theta}$. This comes from the last of the equations of motion:

$$
\begin{aligned}
I\ddot\theta &= \ell\sin\theta\,\lambda_1 - \ell\cos\theta\,\lambda_2 \\
&= -\tfrac{Mm}{M+m}\,\ell^2\big(\sin^2\!\theta\,\ddot\theta + \sin\theta\cos\theta\,\dot\theta^2\big) - mg\ell\cos\theta - m\ell^2\big(\cos^2\!\theta\,\ddot\theta - \sin\theta\cos\theta\,\dot\theta^2\big) \\
&= -mg\ell\cos\theta - m\ell^2\Big(1 - \tfrac{m}{M+m}\,\sin^2\!\theta\Big)\,\ddot\theta + \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot\theta^2 \ . \qquad (17.59)
\end{aligned}
$$

Collecting terms proportional to $\ddot\theta$, we obtain

$$
\big(I + m\ell^2 - \tfrac{m}{M+m}\,\sin^2\!\theta\big)\,\ddot\theta = \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot\theta^2 - mg\ell\,\cos\theta \ . \qquad (17.60)
$$

We are now ready to demand $Q_x = \lambda_1 = 0$, which entails

$$
\ddot\theta = -\frac{\cos\theta}{\sin\theta}\,\dot\theta^2 \ . \qquad (17.61)
$$

Substituting this into eqn. 17.60, we obtain

$$
\big(I + m\ell^2\big)\,\dot\theta^2 = mg\ell\,\sin\theta \ . \qquad (17.62)
$$

Finally, we substitute this into eqn. 17.51 to obtain an equation for the detachment angle, θ^*

$$
E - \frac{\Lambda^2}{2(M+m)} = \left(3 - \frac{m}{M+m}\cdot\frac{m\ell^2}{I+m\ell^2}\,\sin^2\!\theta^*\right)\cdot\tfrac{1}{2}mg\ell\,\sin\theta^* \ . \qquad (17.63)
$$

If our initial conditions are that the system starts from rest[1] with an angle of inclination θ_0, then the detachment condition becomes

$$
\begin{aligned}
\sin\theta_0 &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\Big(\tfrac{m}{M+m}\Big)\Big(\tfrac{m\ell^2}{I+m\ell^2}\Big)\sin^3\!\theta^* \\
&= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\,\alpha^{-1}\,\sin^3\!\theta^* \ , \qquad (17.64)
\end{aligned}
$$

where

$$
\alpha \equiv \left(1 + \frac{M}{m}\right)\left(1 + \frac{I}{m\ell^2}\right) \ . \qquad (17.65)
$$

Note that $\alpha \geq 1$, and that when $M/m = \infty$[2], we recover $\theta^* = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big)$. For finite α, the ladder detaches at a larger value of θ^*. A sketch of θ^* *versus* θ_0 is provided in Fig. 17.5. Note that, provided $\alpha \geq 1$, detachment always occurs for some unique value θ^* for each θ_0.

[1] 'Rest' means that the initial velocities are $\dot X = 0$ and $\dot\theta = 0$, and hence $\Lambda = 0$ as well.

[2] I must satisfy $I \leq m\ell^2$.

17.3 F05 Physics 110A Final Exam

[**1**] Two blocks and three springs are configured as in Fig. 17.6. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

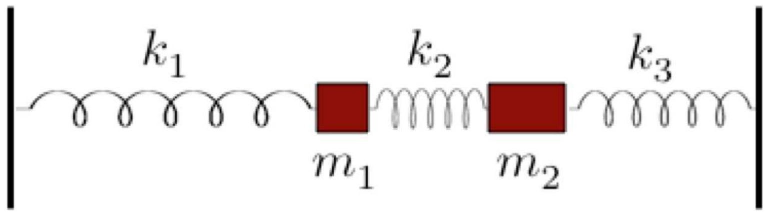

Figure 17.6: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.
`[5 points]`

(b) Find the T and V matrices.
`[5 points]`

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.
`[5 points]`

(d) Find the normal modes of oscillation.
`[5 points]`

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.
`[5 points]`

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2} m_1 \, \dot{x}_1^2 + \tfrac{1}{2} m_2 \, \dot{x}_2^2 - \tfrac{1}{2} k_1 \, x_1^2 - \tfrac{1}{2} k_2 \, (x_2 - x_1)^2 - \tfrac{1}{2} k_3 \, x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i \, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda\,\omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} .$$

The determinant is

$$\begin{aligned} \det\,(\omega^2 \mathrm{T} - \mathrm{V}) &= (2\lambda^2 - 11\lambda + 14)\,k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\,k^2 \ . \end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V})\,\vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)}\,\mathrm{T}_{ij}\,\psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \qquad , \qquad \mathcal{C}_2 = \frac{1}{\sqrt{6m}} \ . \tag{17.66}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \ .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 \ .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) \ . \end{aligned}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega\_t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega\_t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

[2] Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) ,$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?
`[7 points]`

(c) For the case where a circular orbit exists, sketch the phase curves for the radial motion in the $(r, \dot{r})$ half-plane. Identify the energy ranges for bound and unbound orbits.
`[5 points]`

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $|\eta| \ll r_0$. Find the equation for the shape $\eta(\phi)$ of the perturbation.
`[8 points]`

(d) What is the angle $\Delta\phi$ through which periapsis changes each cycle? For which value(s) of ℓ does the perturbed orbit not precess?
`[5 points]`

Solution

(a) The effective potential is

$$\begin{aligned} U_{\text{eff}}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^2} + U_0 \ln\left(\frac{r^2}{r^2+b^2}\right) . \end{aligned}$$

where $\mu = m_1 m_2/(m_1 + m_1)$ is the reduced mass. For a circular orbit, we must have $U'_{\text{eff}}(r) = 0$, or

$$\frac{l^2}{\mu r^3} = U'(r) = \frac{2rU_0 b^2}{r^2\,(r^2+b^2)} .$$

The solution is

$$\boxed{r_0^2 = \frac{b^2\ell^2}{2\mu b^2 U_0 - \ell^2}}$$

Since $r_0^2 > 0$, the condition on ℓ is

$$\boxed{\ell < \ell_{\text{c}} \equiv \sqrt{2\mu b^2 U_0}}$$

For large r, we have

$$U_{\text{eff}}(r) = \left(\frac{\ell^2}{2\mu} - U_0\, b^2\right) \cdot \frac{1}{r^2} + \mathcal{O}(r^{-4}) .$$

Thus, for $\ell < \ell_{\text{c}}$ the effective potential is negative for sufficiently large values of r. Thus, over the range $\ell < \ell_{\text{c}}$, we must have $U_{\text{eff,min}} < 0$, which must be a global minimum, since $U_{\text{eff}}(0^+) = \infty$ and $U_{\text{eff}}(\infty) = 0$. Therefore, the circular orbit is stable whenever it exists.

(b) Let $\ell = \epsilon\,\ell_{\text{c}}$. The effective potential is then

$$U_{\text{eff}}(r) = U_0\, f(r/b) ,$$

where the dimensionless effective potential is

$$f(s) = \frac{\epsilon^2}{s^2} - \ln(1 + s^{-2}) .$$

The phase curves are plotted in Fig. 17.7.

(c) The energy is

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\text{eff}}(r) , \end{aligned}$$

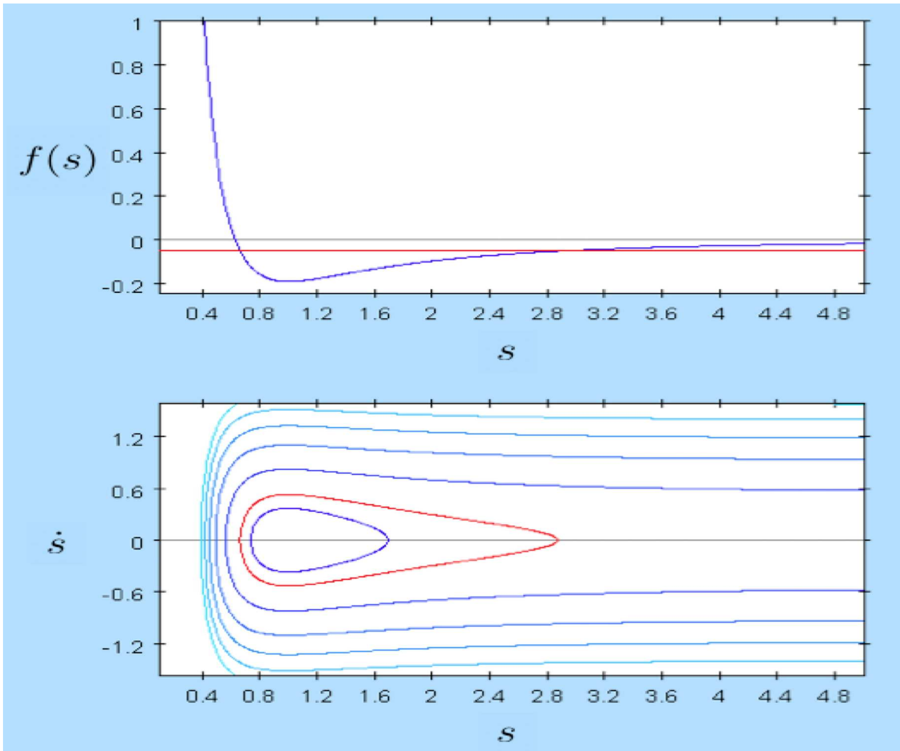

Figure 17.7: Phase curves for the scaled effective potential $f(s) = \epsilon\, s^{-2} - \ln(1 + s^{-2})$, with $\epsilon = \frac{1}{\sqrt{2}}$. Here, $\epsilon = \ell/\ell_c$. The dimensionless time variable is $\tau = t \cdot \sqrt{U_0/mb^2}$.

where we've used $\dot{r} = \dot{\phi}\, r'$ along with $\ell = \mu r^2 \dot{\phi}$. Writing $r = r_0 + \eta$ and differentiating E with respect to ϕ, we find

$$\eta'' = -\beta^2 \eta \qquad , \qquad \beta^2 = \frac{\mu r_0^4}{\ell^2}\, U''_{\rm eff}(r_0)\ .$$

For our potential, we have

$$\boxed{\beta^2 = 2 - \frac{\ell^2}{\mu b^2 U_0} = 2\left(1 - \frac{\ell^2}{\ell_c^2}\right)}$$

The solution is

$$\boxed{\eta(\phi) = A\, \cos(\beta\phi + \delta)} \qquad (17.67)$$

where A and δ are constants.

(d) The change of periapsis per cycle is

$$\boxed{\Delta\phi = 2\pi\big(\beta^{-1} - 1\big)}$$

If $\beta > 1$ then $\Delta\phi < 0$ and periapsis *advances* each cycle (*i.e.*it comes sooner with every cycle). If $\beta < 1$ then $\Delta\phi > 0$ and periapsis *recedes*. For $\beta = 1$, which means $\ell = \sqrt{\mu b^2 U_0}$, there is no precession and $\Delta\phi = 0$.

[3] A particle of charge e moves in three dimensions in the presence of a uniform magnetic field $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$ and a uniform electric field $\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}$. The potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = -e\,E_0\,x - \frac{e}{c}\,B_0\,x\,\dot{y} \ ,$$

where we have chosen the gauge $\boldsymbol{A} = B_0\,x\,\hat{\boldsymbol{y}}$.

(a) Find the canonical momenta p_x, p_y, and p_z.
`[7 points]`

(b) Identify all conserved quantities.
`[8 points]`

(c) Find a complete, general solution for the motion of the system $\{x(t), y(t), x(t)\}$.
`[10 points]`

Solution

(a) The Lagrangian is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}\,B_0\,x\,\dot{y} + e\,E_0\,x \ .$$

The canonical momenta are

$$\boxed{p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}}$$

$$\boxed{p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{e}{c}\,B_0\,x}$$

$$\boxed{p_x = \frac{\partial L}{\partial \dot{z}} = m\dot{z}}$$

(b) There are three conserved quantities. First is the momentum p_y, since $F_y = \frac{\partial L}{\partial y} = 0$. Second is the momentum p_z, since $F_y = \frac{\partial L}{\partial z} = 0$. The third conserved quantity is the Hamiltonian, since $\frac{\partial L}{\partial t} = 0$. We have

$$H = p_x\,\dot{x} + p_y\,\dot{y} + p_z\,\dot{z} - L$$

$$\Rightarrow \quad \boxed{H = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\,E_0\,x}$$

(c) The equations of motion are

$$\begin{aligned}\ddot{x} - \omega_c\,\dot{y} &= \frac{e}{m}\,E_0 \\ \ddot{y} + \omega_c\,\dot{x} &= 0 \\ \ddot{z} &= 0\ .\end{aligned}$$

The second equation can be integrated once to yield $\dot{y} = \omega_c(x_0 - x)$, where x_0 is a constant. Substituting this into the first equation gives

$$\ddot{x} + \omega_c^2\,x = \omega_c^2\,x_0 + \frac{e}{m}\,E_0\ .$$

This is the equation of a constantly forced harmonic oscillator. We can therefore write the general solution as

$$\boxed{x(t) = x_0 + \frac{eE_0}{m\omega_c^2} + A\,\cos\big(\omega_c t + \delta\big)}$$

$$\boxed{y(t) = y_0 - \frac{eE_0}{m\omega_c}\,t - A\,\sin\big(\omega_c t + \delta\big)}$$

$$\boxed{z(t) = z_0 + \dot{z}_0\,t}$$

Note that there are six constants, $\big\{A,\ \delta,\ x_0,\ y_0,\ z_0,\ \dot{z}_0\big\}$, are are required for the general solution of three coupled second order ODEs.

[4] An $N = 1$ dynamical system obeys the equation

$$\frac{du}{dt} = ru + 2bu^2 - u^3\ ,$$

where r is a control parameter, and where $b > 0$ is a constant.

(a) Find and classify all bifurcations for this system.
`[7 points]`

(b) Sketch the fixed points u^* *versus* r.
`[6 points]`

Now let $b = 3$. At time $t = 0$, the initial value of u is $u(0) = 1$. The control parameter r is then increased *very slowly* from $r = -20$ to $r = +20$, and then decreased very slowly back down to $r = -20$.

(c) What is the value of u when $r = -5$ on the *increasing* part of the cycle?
`[3 points]`

(d) What is the value of u when $r = +16$ on the *increasing* part of the cycle?
[3 points]

(e) What is the value of u when $r = +16$ on the *decreasing* part of the cycle?
[3 points]

(f) What is the value of u when $r = -5$ on the *decreasing* part of the cycle?
[3 points]

Solution

(a) Setting $\dot{u} = 0$ we obtain

$$(u^2 - 2bu - r)\, u = 0 \ .$$

The roots are

$$u = 0 \quad , \quad u = b \pm \sqrt{b^2 + r} \ .$$

The roots at $u = u_\pm = b \pm \sqrt{b^2 + r}$ are only present when $r > -b^2$. At $r = -b^2$ there is a *saddle-node bifurcation*. The fixed point $u = u_-$ crosses the fixed point at $u = 0$ at $r = 0$, at which the two fixed points exchange stability. This corresponds to a *transcritical bifurcation*. In Fig. 17.8 we plot $\dot{u}/b^3$ *versus* u/b for several representative values of r/b^2. Note that, defining $\tilde{u} = u/b$, $\tilde{r} = r/b^2$, and $\tilde{t} = b^2 t$ that our $N = 1$ system may be written

$$\frac{d\tilde{u}}{d\tilde{t}} = \left(\tilde{r} + 2\tilde{u} - \tilde{u}^2\right) \tilde{u} \ ,$$

which shows that it is only the dimensionless combination $\tilde{r} = r/b^2$ which enters into the location and classification of the bifurcations.

(b) A sketch of the fixed points u^* *versus* r is shown in Fig. 17.9. Note the two bifurcations at $r = -b^2$ (saddle-node) and $r = 0$ (transcritical).

(c) For $r = -20 < -b^2 = -9$, the initial condition $u(0) = 1$ flows directly toward the stable fixed point at $u = 0$. Since the approach to the FP is asymptotic, u remains slightly positive even after a long time. When $r = -5$, the FP at $u = 0$ is still stable. *Answer:* $\underline{u = 0}$.

(d) As soon as r becomes positive, the FP at $u^* = 0$ becomes unstable, and u flows to the upper branch u_+. When $r = 16$, we have $u = 3 + \sqrt{3^2 + 16} = 8$. *Answer:* $\underline{u = 8}$.

(e) Coming back down from larger r, the upper FP branch remains stable, thus, $u = 8$ at $r = 16$ on the way down as well. *Answer:* $\underline{u = 8}$.

(f) Now when r first becomes negative on the way down, the upper branch u_+ remains stable. Indeed it remains stable all the way down to $r = -b^2$, the location of the saddle-node bifurcation, at which point the solution $u = u_+$ simply vanishes and the flow is toward $u = 0$ again. Thus, for $r = -5$ on the way down, the system remains on the upper branch, in which case $u = 3 + \sqrt{3^2 - 5} = 5$. *Answer:* $\underline{u = 5}$.

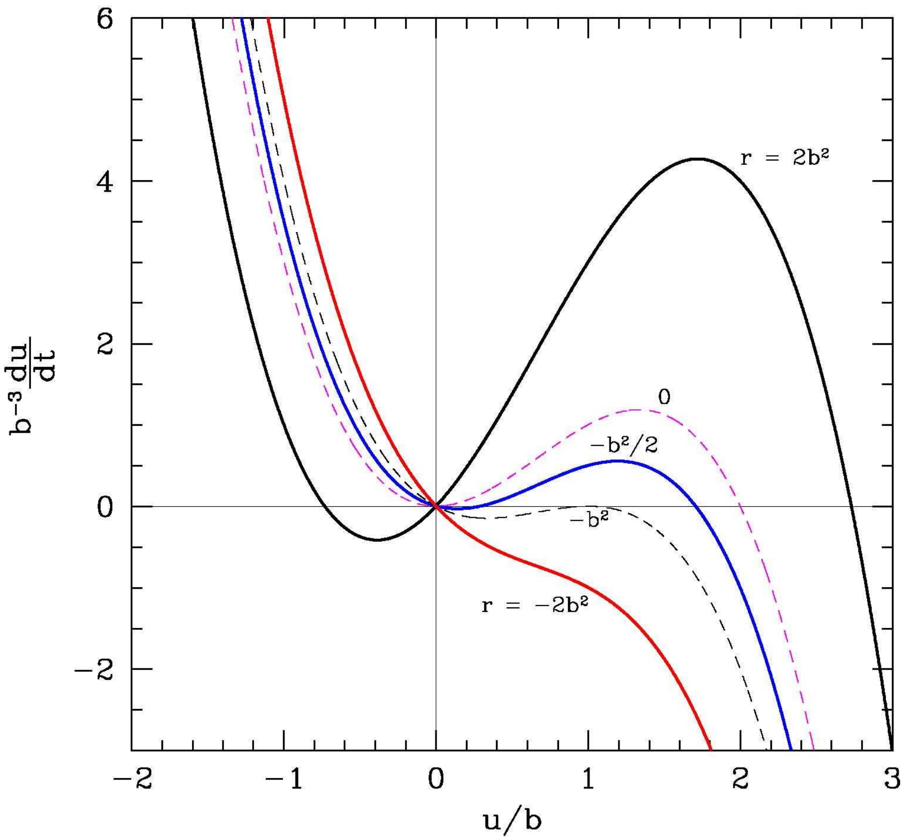

Figure 17.8: Plot of dimensionless 'velocity' $\dot{u}/b^3$ *versus* dimensionless 'coordinate' u/b for several values of the dimensionless control parameter $\tilde{r} = r/b^2$.

17.4 F07 Physics 110A Midterm #1

[**1**] A particle of mass m moves in the one-dimensional potential

$$U(x) = \frac{U_0}{a^4}\left(x^2 - a^2\right)^2 \,. \tag{17.68}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.
`[15 points]`

Solution : Clearly the minima lie at $x = \pm a$ and there is a local maximum at $x = 0$.

(b) Sketch a representative set of phase curves. Be sure to sketch any separatrices which exist, and identify their energies. Also sketch all the phase curves for motions with total energy $E = \frac{1}{2}U_0$. Do the same for $E = 2U_0$.
`[15 points]`

Solution : See Fig. 17.10 for the phase curves. Clearly $U(\pm a) = 0$ is the minimum of the

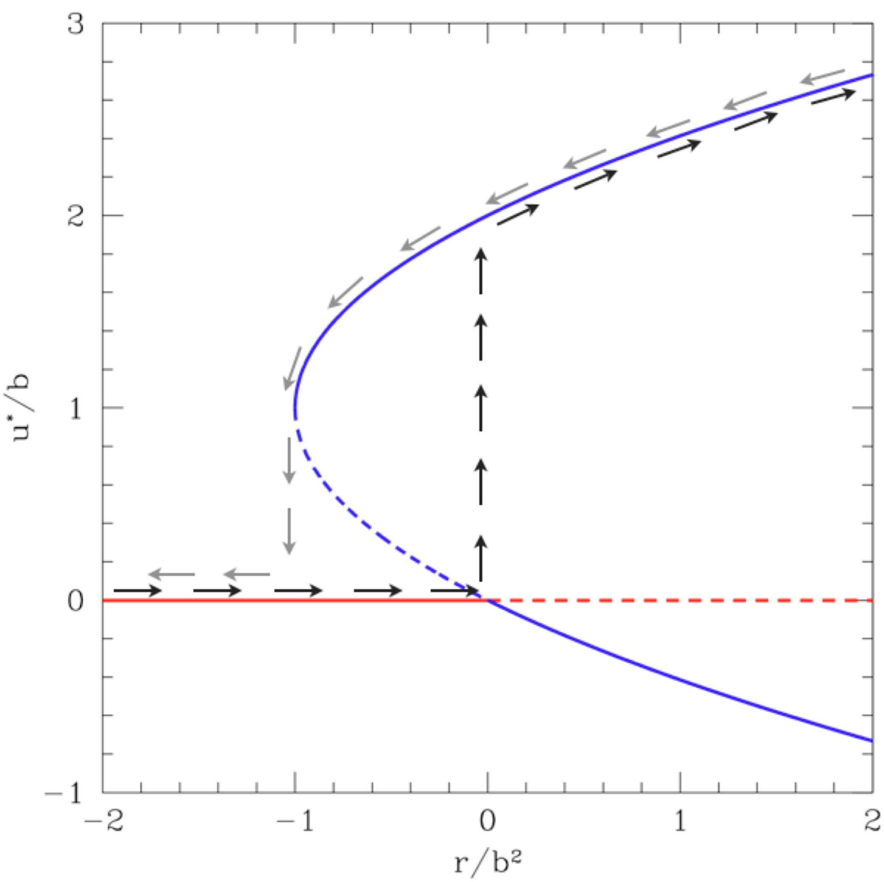

Figure 17.9: Fixed points and their stability *versus* control parameter for the $N = 1$ system $\dot{u} = ru + 2bu^2 - u^3$. Solid lines indicate stable fixed points; dashed lines indicate unstable fixed points. There is a saddle-node bifurcation at $r = -b^2$ and a transcritical bifurcation at $r = 0$. The hysteresis loop in the upper half plane $u > 0$ is shown. For $u < 0$ variations of the control parameter r are reversible and there is no hysteresis.

potential, and $U(0) = U_0$ is the local maximum and the energy of the separatrix. Thus, $E = \frac{1}{2}U_0$ cuts through the potential in both wells, and the phase curves at this energy form two disjoint sets. For $E < U_0$ there are four turning points, at

$$x_{1,<} = -a\sqrt{1 + \sqrt{\frac{E}{U_0}}} \quad , \quad x_{1,>} = -a\sqrt{1 - \sqrt{\frac{E}{U_0}}}$$

and

$$x_{2,<} = a\sqrt{1 - \sqrt{\frac{E}{U_0}}} \quad , \quad x_{2,>} = a\sqrt{1 + \sqrt{\frac{E}{U_0}}}$$

For $E = 2U_0$, the energy is above that of the separatrix, and there are only two turning points, $x_{1,<}$ and $x_{2,>}$. The phase curve is then connected.

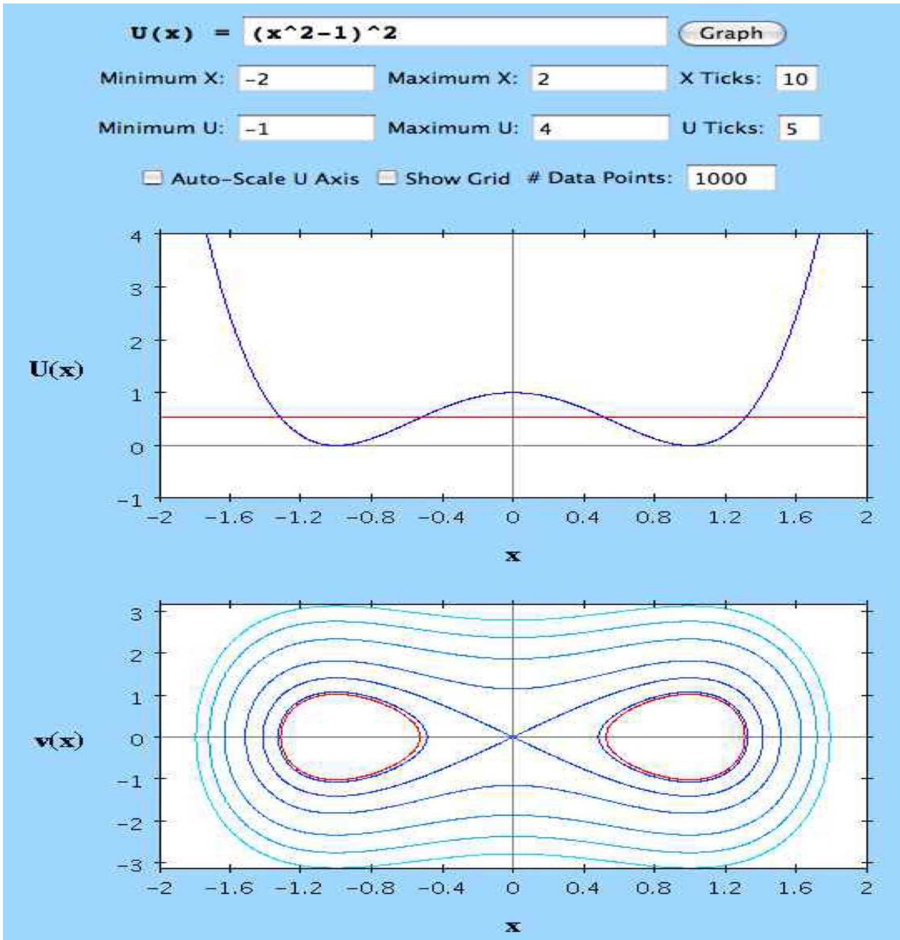

Figure 17.10: Sketch of the double well potential $U(x) = (U_0/a^4)(x^2 - a^2)^2$, here with distances in units of a, and associated phase curves. The separatrix is the phase curve which runs through the origin. Shown in red is the phase curve for $U = \frac{1}{2}\,U_0$, consisting of two deformed ellipses. For $U = 2\,U_0$, the phase curve is connected, lying outside the separatrix.

(c) The phase space dynamics are written as $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. Find the upper and lower components of the vector field $\boldsymbol{V}$.
`[10 points]`

Solution :

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}\,U'(x) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{4U_0}{a^2}\,x\,(x^2 - a^2) \end{pmatrix} . \tag{17.69}$$

(d) Derive and expression for the period T of the motion when the system exhibits small oscillations about a potential minimum.
`[10 points]`

Solution : Set $x = \pm a + \eta$ and Taylor expand:

$$U(\pm a + \eta) = \frac{4U_0}{a^2}\,\eta^2 + \mathcal{O}(\eta^3) \ . \tag{17.70}$$

Equating this with $\frac{1}{2}k\,\eta^2$, we have the effective spring constant $k = 8U_0/a^2$, and the small oscillation frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,U_0}{ma^2}} \ . \tag{17.71}$$

The period is $2\pi/\omega_0$.

[**2**] An *R-L-C* circuit is shown in fig. 17.11. The resistive element is a light bulb. The inductance is $L = 400\,\mu$H; the capacitance is $C = 1\,\mu$F; the resistance is $R = 32\,\Omega$. The voltage $V(t)$ oscillates sinusoidally, with $V(t) = V_0 \cos(\omega t)$, where $V_0 = 4\,$V. In this problem, you may neglect all transients; we are interested in the late time, steady state operation of this circuit. Recall the relevant MKS units:

$$1\,\Omega = 1\,\text{V}\cdot\text{s}\,/\,\text{C} \quad , \quad 1\,\text{F} = 1\,\text{C}\,/\,\text{V} \quad , \quad 1\,\text{H} = 1\,\text{V}\cdot\text{s}^2/\,\text{C} \ .$$

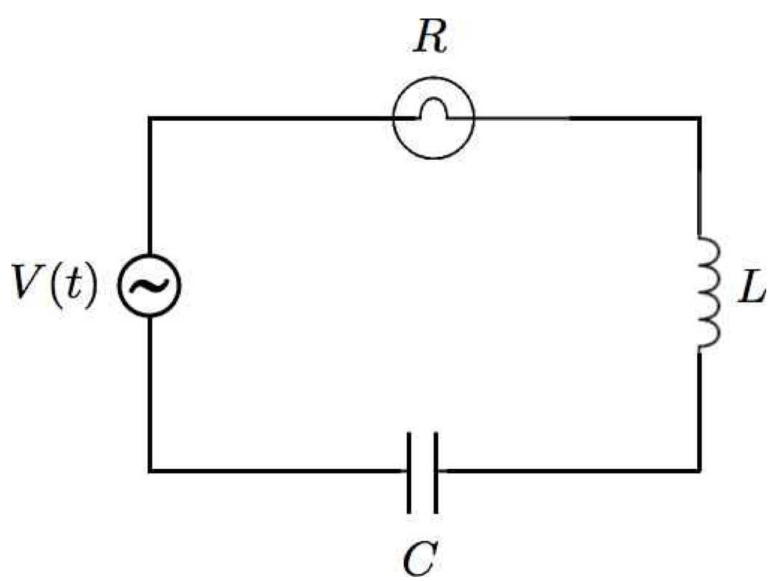

Figure 17.11: An *R-L-C* circuit in which the resistive element is a light bulb.

(a) Is this circuit underdamped or overdamped?
`[10 points]`

Solution : We have

$$\omega_0 = (LC)^{-1/2} = 5\times 10^4\,\text{s}^{-1} \quad , \quad \beta = \frac{R}{2L} = 4\times 10^4\,\text{s}^{-1} \ .$$

Thus, $\omega_0^2 > \beta^2$ and the circuit is *underdamped*.

(b) Suppose the bulb will only emit light when the average power dissipated by the bulb is greater than a threshold $P_{\text{th}} = \frac{2}{9}\,W$. For fixed $V_0 = 4\,\text{V}$, find the frequency range for ω over which the bulb emits light. Recall that the instantaneous power dissipated by a resistor is $P_R(t) = I^2(t)R$. (Average this over a cycle to get the average power dissipated.)
`[20 points]`

Solution : The charge on the capacitor plate obeys the ODE

$$L\,\ddot{Q} + R\,\dot{Q} + \frac{Q}{C} = V(t)\ .$$

The solution is

$$Q(t) = Q_{\text{hom}}(t) + A(\omega)\,\frac{V_0}{L}\,\cos\big(\omega t - \delta(\omega)\big)\ ,$$

with

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\bigg(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\bigg)\ .$$

Thus, ignoring the transients, the power dissipated by the bulb is

$$\begin{aligned} P_R(t) &= \dot{Q}^2(t)\,R \\ &= \omega^2 A^2(\omega)\,\frac{V_0^2 R}{L^2}\,\sin^2\big(\omega t - \delta(\omega)\big)\ . \end{aligned}$$

Averaging over a period, we have $\langle\,\sin^2(\omega t - \delta)\,\rangle = \frac{1}{2}$, so

$$\langle\,P_R\,\rangle = \omega^2 A^2(\omega)\,\frac{V_0^2 R}{2L^2} = \frac{V_0^2}{2\,R}\cdot\frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\ .$$

Now $V_0^2/2R = \frac{1}{4}\,\text{W}$. So $P_{\text{th}} = \alpha V_0^2/2R$, with $\alpha = \frac{8}{9}$. We then set $\langle P_R\rangle = P_{\text{th}}$, whence

$$(1-\alpha)\cdot 4\beta^2\omega^2 = \alpha\,(\omega_0^2 - \omega^2)^2\ .$$

The solutions are

$$\omega = \pm\sqrt{\frac{1-\alpha}{\alpha}}\,\beta + \sqrt{\bigg(\frac{1-\alpha}{\alpha}\bigg)\beta^2 + \omega_0^2} = \big(3\sqrt{3} \pm \sqrt{2}\big) \times 1000\,\text{s}^{-1}\ .$$

(c) Compare the expressions for the instantaneous power dissipated by the voltage source, $P_V(t)$, and the power dissipated by the resistor $P_R(t) = I^2(t)R$. If $P_V(t) \neq P_R(t)$, where does the power extra power go or come from? What can you say about the averages of P_V and $P_R(t)$ over a cycle? Explain your answer.
`[20 points]`

Solution : The instantaneous power dissipated by the voltage source is

$$\begin{aligned} P_V(t) = V(t)\,I(t) &= -\omega\,A\,\frac{V_0}{L}\,\sin(\omega t - \delta)\,\cos(\omega t) \\ &= \omega\,A\,\frac{V_0}{2L}\,\Big(\sin\delta - \sin(2\omega t - \delta)\Big)\ . \end{aligned}$$

As we have seen, the power dissipated by the bulb is

$$P_R(t) = \omega^2 A^2 \, \frac{V_0^2 R}{L^2} \sin^2(\omega t - \delta) \ .$$

These two quantities are not identical, but they do have identical time averages over one cycle:

$$\langle \, P_V(t) \, \rangle = \langle \, P_R(t) \, \rangle = \frac{V_0^2}{2R} \cdot 4\beta^2 \, \omega^2 \, A^2(\omega) \ .$$

Energy conservation means

$$P_V(t) = P_R(t) + \dot{E}(t) \ ,$$

where

$$E(t) = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

is the energy in the inductor and capacitor. Since $Q(t)$ is periodic, the average of $\dot{E}$ over a cycle must vanish, which guarantees $\langle \, P_V(t) \, \rangle = \langle \, P_R(t) \, \rangle$.

What was not asked:

(d) What is the maximum charge $Q_{\rm max}$ on the capacitor plate if $\omega = 3000 \, {\rm s}^{-1}$?
[10 points]

Solution : Kirchoff's law gives for this circuit the equation

$$\ddot{Q} + 2\beta \, \dot{Q} + \omega_0^2 \, Q = \frac{V_0}{L} \, \cos(\omega t) \ ,$$

with the solution

$$Q(t) = Q_{\rm hom}(t) + A(\omega) \, \frac{V_0}{L} \, \cos\big(\omega t - \delta(\omega)\big) \ ,$$

where $Q_{\rm hom}(t)$ is the homogeneous solution, *i.e.* the transient which we ignore, and

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \ .$$

Then

$$Q_{\rm max} = A(\omega) \, \frac{V_0}{L} \ .$$

Plugging in $\omega = 3000 \, {\rm s}^{-1}$, we have

$$A(\omega) = \big[(5^2 - 4^2)^2 + 4 \cdot 4^2 \cdot 3^2\big]^{-1/2} \times 10^{-3} \, {\rm s}^2 = \frac{1}{8\sqrt{13}} \times 10^{-3} \, {\rm s}^2 \ .$$

Since $V_0/L = 10^4 \, {\rm C/s}^2$, we have

$$Q_{\rm max} = \frac{5}{4\sqrt{13}} \, {\rm Coul} \ .$$

17.5 F07 Physics 110A Midterm #2

[**1**] A point mass m slides frictionlessly, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as depicted in fig. 17.12. All motion is within the plane of the figure.

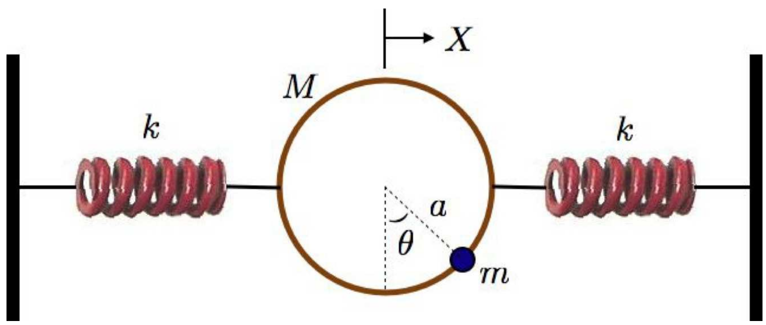

Figure 17.12: A point mass m slides frictionlessly along a massive ring of radius a and mass M, which is affixed by horizontal springs to two fixed vertical surfaces.

`(a)` Choose as generalized coordinates the horizontal displacement X of the center of the ring with respect to equilibrium, and the angle θ a radius to the mass m makes with respect to the vertical (see fig. 17.12). You may assume that at $X = 0$ the springs are both unstretched. Find the Lagrangian $L(X, \theta, \dot{X}, \dot{\theta}, t)$.
`[15 points]`

The coordinates of the mass point are

$$x = X + a \sin\theta \quad , \quad y = -a\cos\theta \ .$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{X} + a\cos\theta\,\dot{\theta}\big)^2 + \tfrac{1}{2}ma^2\sin^2\!\theta\,\dot{\theta}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X}\,\dot{\theta} \ . \end{aligned}$$

The potential energy is

$$U = kX^2 - mga\cos\theta \ .$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X} - kX^2 + mga\cos\theta \ .$$

(b) Find the generalized momenta p_X and p_θ, and the generalized forces F_X and F_θ
`[10 points]`

We have

$$p_X = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + ma\cos\theta\,\dot{\theta} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\cos\theta\,\dot{X} \ .$$

For the forces,

$$F_X = \frac{\partial L}{\partial X} = -2kX \quad , \quad F_\theta = \frac{\partial L}{\partial \theta} = -ma \sin\theta \, \dot{X} \, \dot{\theta} - mga \sin\theta \ .$$

(c) Derive the equations of motion.
`[15 points]`

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} \ ,$$

for each generalized coordinate q_σ. For X we have

$$(M+m)\ddot{X} + ma \cos\theta \, \ddot{\theta} - ma \sin\theta \, \dot{\theta}^2 = -2kX \ .$$

For θ,

$$ma^2 \, \ddot{\theta} + ma \cos\theta \ddot{X} = -mga \sin\theta \ .$$

(d) Find expressions for all conserved quantities.
`[10 points]`

Horizontal and vertical translational symmetries are broken by the springs and by gravity, respectively. The remaining symmetry is that of time translation. From $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$, we have that $H = \sum_\sigma p_\sigma \, \dot{q}_\sigma - L$ is conserved. For this problem, the kinetic energy is a homogeneous function of degree 2 in the generalized velocities, and the potential is velocity-independent. Thus,

$$H = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma \cos\theta \, \dot{X} \, \dot{\theta} + kX^2 - mga \cos\theta \ .$$

[2] A point particle of mass m moves in three dimensions in a helical potential

$$U(\rho,\phi,z) = U_0\,\rho\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) .$$

We call b the pitch of the helix.

(a) Write down the Lagrangian, choosing (ρ,ϕ,z) as generalized coordinates.
`[10 points]`

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) - U_0\,\rho\,\cos\left(\phi - \frac{2\pi z}{b}\right)$$

(b) Find the equations of motion.
`[20 points]`

Clearly

$$p_\rho = m\dot\rho \quad , \quad p_\phi = m\rho^2\,\dot\phi \quad , \quad p_z = m\dot z \ ,$$

and

$$F_\rho = m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_\phi = U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_z = -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) .$$

Thus, the equation of motion are

$$\begin{aligned} m\ddot\rho &= m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\rho^2\,\ddot\phi + 2m\rho\,\dot\rho\,\dot\phi &= U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\ddot z &= -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) . \end{aligned}$$

(c) Show that there exists a continuous one-parameter family of coordinate transformations which leaves L invariant. Find the associated conserved quantity, Λ. Is anything else conserved?
`[20 points]`

Due to the helical symmetry, we have that

$$\phi \to \phi + \zeta \quad , \quad z \to z + \frac{b}{2\pi}\,\zeta$$

is such a continuous one-parameter family of coordinate transformations. Since it leaves the combination $\phi - \frac{2\pi z}{b}$ unchanged, we have that $\frac{dL}{d\zeta} = 0$, and

$$\begin{aligned}
\Lambda &= p_\rho \, \frac{\partial \rho}{\partial \zeta}\bigg|_{\zeta=0} + p_\phi \, \frac{\partial \phi}{\partial \zeta}\bigg|_{\zeta=0} + p_z \, \frac{\partial z}{\partial \zeta}\bigg|_{\zeta=0} \\
&= p_\phi + \frac{b}{2\pi} \, p_z \\
&= m\rho^2 \, \dot\phi + \frac{mb}{2\pi} \, \dot z
\end{aligned}$$

is the conserved Noether 'charge'. The other conserved quantity is the Hamiltonian,

$$H = \tfrac{1}{2} m\big(\dot\rho^2 + \rho^2 \dot\phi^2 + \dot z^2\big) + U_0 \, \rho \, \cos\left(\phi - \frac{2\pi z}{b}\right) .$$

Note that $H = T + U$, because T is homogeneous of degree 2 and U is homogeneous of degree 0 in the generalized velocities.

17.6 F07 Physics 110A Final Exam

[1] Two masses and two springs are configured linearly and externally driven to rotate with angular velocity ω about a fixed point on a horizontal surface, as shown in fig. 17.13. The unstretched length of each spring is a.

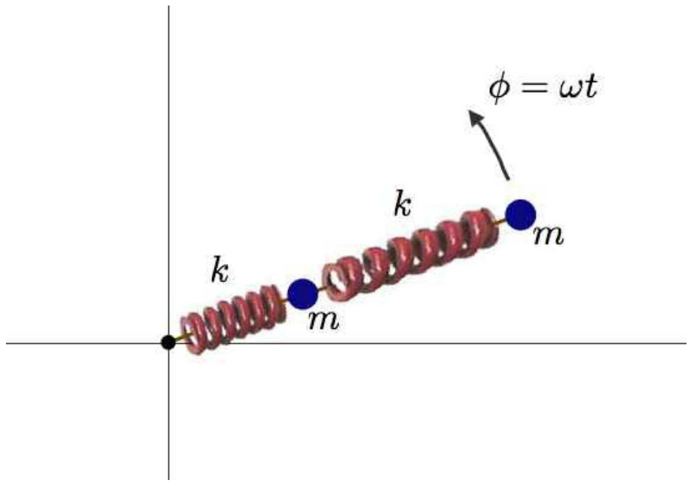

Figure 17.13: Two masses and two springs rotate with angular velocity ω.

(a) Choose as generalized coordinates the radial distances $r_{1,2}$ from the origin. Find the Lagrangian $L(r_1, r_2, \dot{r}_1, \dot{r}_2, t)$.
[5 points]

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2\, r_1^2 + \omega^2\, r_2^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.72)$$

(b) Derive expressions for all conserved quantities.
[5 points]

The Hamiltonian is conserved. Since the kinetic energy is not homogeneous of degree 2 in the generalized velocities, $H \neq T + U$. Rather,

$$H = \sum_\sigma p_\sigma\, \dot{q}_\sigma - L \qquad (17.73)$$

$$= \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2\big) - \tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.74)$$

We could define an effective potential

$$U_{\rm eff}(r_1, r_2) = -\tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.75)$$

Note the first term, which comes from the kinetic energy, has an interpretation of a fictitious potential which generates a *centrifugal* force.

(c) What equations determine the equilibrium radii r_1^0 and r_2^0? (You do not have to solve these equations.)
[5 points]

The equations of equilibrium are $F_\sigma = 0$. Thus,

$$0 = F_1 = \frac{\partial L}{\partial r_1} = m\omega^2 r_1 - k\,(r_1 - a) + k\,(r_2 - r_1 - a) \tag{17.76}$$

$$0 = F_2 = \frac{\partial L}{\partial r_2} = m\omega^2 r_2 - k\,(r_2 - r_1 - a)\ . \tag{17.77}$$

(d) Suppose now that the system is not externally driven, and that the angular coordinate ϕ is a dynamical variable like r_1 and r_2. Find the Lagrangian $L(r_1, r_2, \phi, \dot{r}_1, \dot{r}_2, \dot{\phi}, t)$.
[5 points]

Now we have

$$L = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\,\dot{\phi}^2 + r_2^2\,\dot{\phi}^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{17.78}$$

(e) For the system described in part (d), find expressions for all conserved quantities.
[5 points]

There are two conserved quantities. One is p_ϕ, owing to the fact the ϕ is cyclic in the Lagrangian. *I.e.* $\phi \to \phi + \zeta$ is a continuous one-parameter coordinate transformation which leaves L invariant. We have

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\big(r_1^2 + r_2^2\big)\,\dot{\phi}\ . \tag{17.79}$$

The second conserved quantity is the Hamiltonian, which is now $H = T + U$, since T is homogeneous of degree 2 in the generalized velocities. Using conservation of momentum, we can write

$$H = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2\big) + \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{17.80}$$

Once again, we can define an effective potential,

$$U_{\text{eff}}(r_1, r_2) = \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ , \tag{17.81}$$

which is different than the effective potential from part (b). However in both this case and in part (b), we have that the radial coordinates obey the equations of motion

$$m\ddot{r}_j = -\frac{\partial U_{\text{eff}}}{\partial r_j}\ , \tag{17.82}$$

for $j = 1, 2$. Note that this equation of motion follows directly from $\dot{H} = 0$.

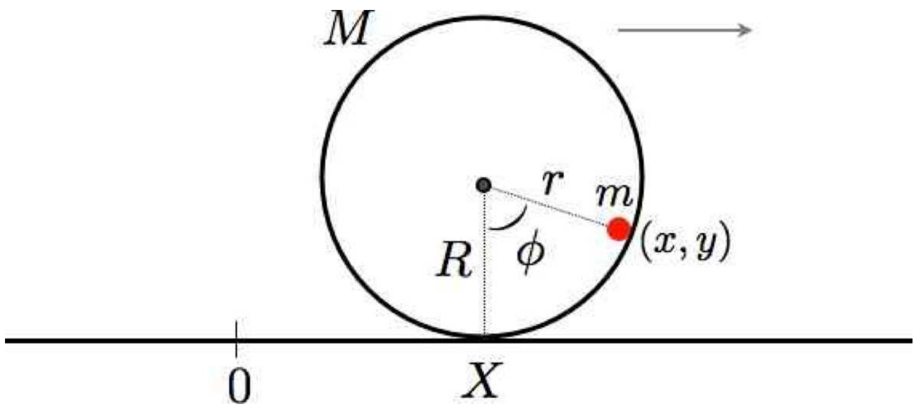

Figure 17.14: A mass point m rolls inside a hoop of mass M and radius R which rolls without slipping on a horizontal surface.

[2] A point mass m slides inside a hoop of radius R and mass M, which itself rolls without slipping on a horizontal surface, as depicted in fig. 17.14.

Choose as general coordinates (X, ϕ, r), where X is the horizontal location of the center of the hoop, ϕ is the angle the mass m makes with respect to the vertical ($\phi = 0$ at the bottom of the hoop), and r is the distance of the mass m from the center of the hoop. Since the mass m slides inside the hoop, there is a constraint:

$$G(X, \phi, r) = r - R = 0 \ .$$

Nota bene: The kinetic energy of the moving hoop, including translational and rotational components (but not including the mass m), is $T_{\text{hoop}} = M\dot{X}^2$ (*i.e.* twice the translational contribution alone).

(a) Find the Lagrangian $L(X, \phi, r, \dot{X}, \dot{\phi}, \dot{r}, t)$.
[5 points]

The Cartesian coordinates and velocities of the mass m are

$$x = X + r\sin\phi \qquad \dot{x} = \dot{X} + \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{17.83}$$

$$y = R - r\cos\phi \qquad \dot{y} = -\dot{r}\cos\phi + r\dot{\phi}\sin\phi \tag{17.84}$$

The Lagrangian is then

$$L = \overbrace{(M + \tfrac{1}{2}m)\dot{X}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + m\dot{X}(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)}^{T} - \overbrace{mg(R - r\cos\phi)}^{U} \tag{17.85}$$

Note that we are not allowed to substitute $r = R$ and hence $\dot{r} = 0$ in the Lagrangian *prior* to obtaining the equations of motion. Only *after* the generalized momenta and forces are computed are we allowed to do so.

(b) Find *all* the generalized momenta p_σ, the generalized forces F_σ, and the forces of constraint Q_σ.
[10 points]

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r + m\dot X \sin\phi \tag{17.86}$$

$$p_X = \frac{\partial L}{\partial \dot X} = (2M+m)\dot X + m\dot r \sin\phi + mr\dot\phi \cos\phi \tag{17.87}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2\dot\phi + mr\dot X \cos\phi \tag{17.88}$$

The generalized forces and the forces of constraint are

$$F_r = \frac{\partial L}{\partial r} = mr\dot\phi^2 + m\dot X\dot\phi \cos\phi + mg\cos\phi \qquad Q_r = \lambda\,\frac{\partial G}{\partial r} = \lambda \tag{17.89}$$

$$F_X = \frac{\partial L}{\partial X} = 0 \qquad Q_X = \lambda\,\frac{\partial G}{\partial X} = 0 \tag{17.90}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = m\dot X\dot r \cos\phi - m\dot X\dot\phi \sin\phi - mgr\sin\phi \qquad Q_\phi = \lambda\,\frac{\partial G}{\partial \phi} = 0\ . \tag{17.91}$$

The equations of motion are

$$\dot p_\sigma = F_\sigma + Q_\sigma\ . \tag{17.92}$$

At this point, we can legitimately invoke the constraint $r = R$ and set $\dot r = 0$ in all the p_σ and F_σ.

(c) Derive expressions for all conserved quantities.
[5 points]

There are two conserved quantities, which each derive from continuous invariances of the Lagrangian *which respect the constraint*. The first is the total momentum p_X:

$$F_X = 0 \quad \Longrightarrow \quad P \equiv p_X = \text{constant}\ . \tag{17.93}$$

The second conserved quantity is the Hamiltonian, which in this problem turns out to be the total energy $E = T + U$. Incidentally, we can use conservation of P to write the energy in terms of the variable ϕ alone. From

$$\dot X = \frac{P}{2M+m} - \frac{mR\cos\phi}{2M+m}\,\dot\phi\ , \tag{17.94}$$

we obtain

$$\begin{aligned} E &= \tfrac{1}{2}(2M+m)\dot X^2 + \tfrac{1}{2}mR^2\dot\phi^2 + mR\dot X\dot\phi\cos\phi + mgR(1-\cos\phi) \\ &= \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2}mR^2\left(\frac{1+\alpha\sin^2\!\phi}{1+\alpha}\right)\dot\phi^2 + mgR(1-\cos\phi)\ , \end{aligned} \tag{17.95}$$

where we've defined the dimensionless ratio $\alpha \equiv m/2M$. It is convenient to define the quantity

$$\varOmega^2 \equiv \left(\frac{1+\alpha\sin^2\!\phi}{1+\alpha}\right)\dot\phi^2 + 2\omega_0^2(1-\cos\phi)\ , \tag{17.96}$$

with $\omega_0 \equiv \sqrt{g/R}$. Clearly Ω^2 is conserved, as it is linearly related to the energy E:

$$E = \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2} m R^2 \Omega^2 \ . \tag{17.97}$$

(d) Derive a differential equation of motion involving the coordinate $\phi(t)$ alone. *I.e.* your equation should not involve r, X, or the Lagrange multiplier λ.
[5 points]

From conservation of energy,

$$\frac{d(\Omega^2)}{dt} = 0 \quad \Longrightarrow \quad \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\ddot{\phi} + \left(\frac{\alpha\,\sin\phi\,\cos\phi}{1+\alpha}\right)\dot{\phi}^2 + \omega_0^2\,\sin\phi = 0 \ , \tag{17.98}$$

again with $\alpha = m/2M$. Incidentally, one can use these results in eqns. 17.96 and 17.98 to eliminate $\dot{\phi}$ and $\ddot{\phi}$ in the expression for the constraint force, $Q_r = \lambda = \dot{p}_r - F_r$. One finds

$$\begin{aligned} \lambda &= -mR\,\frac{\dot{\phi}^2 + \omega_0^2\,\cos\phi}{1+\alpha\sin^2\!\phi} \\ &= -\frac{mR\,\omega_0^2}{(1+\alpha\sin^2\!\phi)^2}\left\{(1+\alpha)\left(\frac{\Omega^2}{\omega_0^2} - 4\,\sin^2(\tfrac{1}{2}\phi)\right) + (1+\alpha\sin^2\!\phi)\,\cos\phi\right\} \ . \end{aligned} \tag{17.99}$$

This last equation can be used to determine the angle of detachment, where λ vanishes and the mass m falls off the inside of the hoop. This is because the hoop can only supply a repulsive normal force to the mass m. This was worked out in detail in my lecture notes on constrained systems.

[**3**] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\,\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|^{1/4}$.

(a) Sketch the effective potential $U_{\rm eff}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.
[5 points]

The effective potential is

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + k r^n \tag{17.100}$$

with $n = \frac{1}{4}$. In sketching the effective potential, I have rendered it in dimensionless form,

$$U_{\rm eff}(r) = E_0\,\mathcal{U}_{\rm eff}(r/r_0) \ , \tag{17.101}$$

where $r_0 = (\ell^2/nk\mu)^{(n+2)^{-1}}$ and $E_0 = \big(\frac{1}{2} + \frac{1}{n}\big)\ell^2/\mu r_0^2$, which are obtained from the results of part (b). One then finds

$$\mathcal{U}_{\rm eff}(x) = \frac{n\,x^{-2} + 2\,x^n}{n+2} \ . \tag{17.102}$$

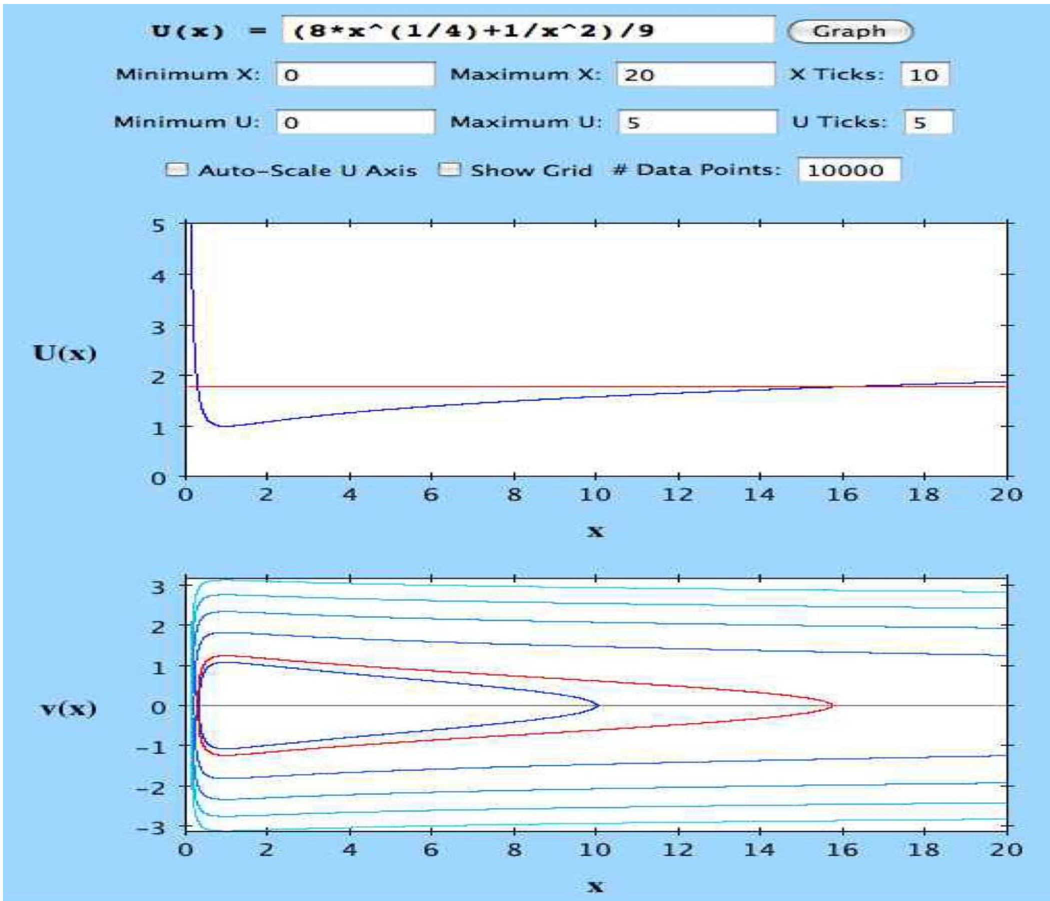

Figure 17.15: The effective $U_{\rm eff}(r) = E_0\, \mathcal{U}_{\rm eff}(r/r_0)$, where r_0 and E_0 are the radius and energy of the circular orbit.

Although it is not obvious from the detailed sketch in fig. 17.15, the effective potential does diverge, albeit slowly, for $r \to \infty$. Clearly it also diverges for $r \to 0$. Thus, the relative coordinate motion is bounded for all energies; the allowed energies are $E \geq E_0$.

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?
[5 points]

For the general power law potential $U(r) = kr^n$, with $nk > 0$ (attractive force), setting $U'_{\rm eff}(r_0) = 0$ yields

$$-\frac{\ell^2}{\mu r_0^3} + nkr_0^{n-1} = 0 \ . \tag{17.103}$$

Thus,

$$r_0 = \left(\frac{\ell^2}{nk\mu}\right)^{\frac{1}{n+2}} = \left(\frac{4\ell^2}{k\mu}\right)^{\frac{4}{9}} \ . \tag{17.104}$$

The orbit $r(t) = r_0$ is stable because the effective potential has a local minimum at $r = r_0$, *i.e.* $U''_{\rm eff}(r_0) > 0$. This is obvious from inspection of the graph of $U_{\rm eff}(r)$ but can also be computed explicitly:

$$\begin{aligned} U''_{\rm eff}(r_0) &= \frac{3\ell^2}{\mu r_0^4} + n(n-1)k r_0^n \\ &= (n+2)\,\frac{\ell^2}{\mu r_0^4}\ . \end{aligned} \tag{17.105}$$

Thus, provided $n > -2$ we have $U''_{\rm eff}(r_0) > 0$.

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?
[5 points]

From the radial coordinate equation $\mu\ddot{r} = -U'_{\rm eff}(r)$, we expand $r = r_0 + \eta$ and find

$$\mu\ddot{\eta} = -U''_{\rm eff}(r_0)\,\eta + \mathcal{O}(\eta^2)\ . \tag{17.106}$$

The radial oscillation frequency is then

$$\omega = (n+2)^{1/2}\,\frac{\ell}{\mu r_0^2} = (n+2)^{1/2}\,n^{\frac{2}{n+2}}\,k^{\frac{2}{n+2}}\,\mu^{-\frac{n}{n+2}}\,\ell^{\frac{n-2}{n+2}}\ . \tag{17.107}$$

The ℓ dependence is what is key here. Clearly

$$\frac{\omega'}{\omega} = \left(\frac{\ell'}{\ell}\right)^{\frac{n-2}{n+2}}\ . \tag{17.108}$$

In our case, with $n = \frac{1}{4}$, we have $\omega \propto \ell^{-7/9}$ and thus

$$\frac{\omega'}{\omega} = 2^{-7/9}\ . \tag{17.109}$$

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.
[5 points]

We have that $\eta(\phi) = \eta_0\cos(\beta\phi + \delta_0)$, with

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu r_0^2}{\ell}\cdot\omega = \sqrt{n+2}\ . \tag{17.110}$$

With $n = \frac{1}{4}$, we have $\beta = \frac{3}{2}$. Thus, the radial coordinate makes three oscillations for every two rotations. The situation is depicted in fig. 17.21.

(e) What value of n would result in a perturbed orbit shaped like that in fig. 17.22?
[5 points]

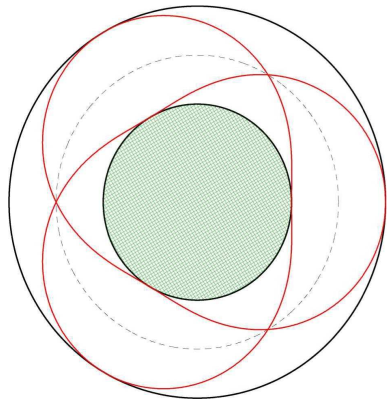

Figure 17.16: Radial oscillations with $\beta = \frac{3}{2}$.

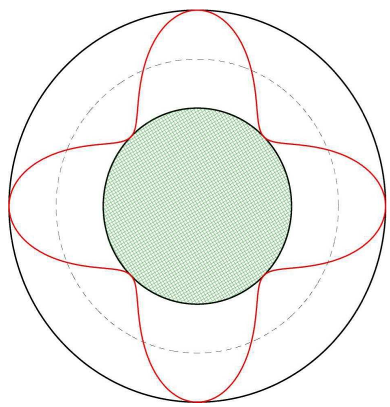

Figure 17.17: Closed precession in a central potential $U(r) = kr^n$.

Clearly $\beta = \sqrt{n+2} = 4$, in order that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$ executes four complete periods over the interval $\phi \in [0, 2\pi]$. This means $n = 14$.

[4] Two masses and three springs are arranged as shown in fig. 17.18. You may assume that in equilibrium the springs are all unstretched with length a. The masses and spring constants are simple multiples of fundamental values, *viz.*

$$m_1 = m \quad , \quad m_2 = 4m \quad , \quad k_1 = k \quad , \quad k_2 = 4k \quad , \quad k_3 = 28k \; . \tag{17.111}$$

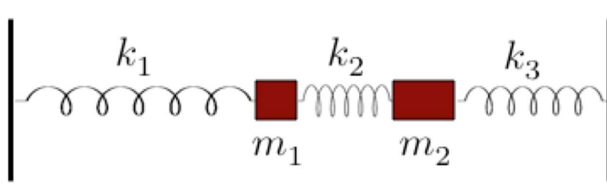

Figure 17.18: Coupled masses and springs.

`(a)` Find the Lagrangian.
`[5 points]`

Choosing displacements relative to equilibrium as our generalized coordinates, we have

$$T = \tfrac{1}{2}m\,\dot\eta_1^2 + 2m\,\dot\eta_2^2 \tag{17.112}$$

and

$$U = \tfrac{1}{2}k\,\eta_1^2 + 2k\,(\eta_2 - \eta_1)^2 + 14k\,\eta_2^2\ . \tag{17.113}$$

Thus,

$$L = T - U = \tfrac{1}{2}m\,\dot\eta_1^2 + 2m\,\dot\eta_2^2 - \tfrac{1}{2}k\,\eta_1^2 - 2k\,(\eta_2 - \eta_1)^2 - 14k\,\eta_2^2\ . \tag{17.114}$$

You are not required to find the equilibrium values of x_1 and x_2. However, suppose all the unstretched spring lengths are a and the total distance between the walls is L. Then, with $x_{1,2}$ being the location of the masses relative to the left wall, we have

$$U = \tfrac{1}{2}k_1\,(x_1 - a)^2 + \tfrac{1}{2}k_2\,(x_2 - x_1 - a)^2 + \tfrac{1}{2}k_3\,(L - x_2 - a)^2\ . \tag{17.115}$$

Differentiating with respect to $x_{1,2}$ then yields

$$\frac{\partial U}{\partial x_1} = k_1\,(x_1 - a) - k_2\,(x_2 - x_1 - a) \tag{17.116}$$

$$\frac{\partial U}{\partial x_2} = k_2\,(x_2 - x_1 - a) - k_3\,(L - x_2 - a)\ . \tag{17.117}$$

Setting these both to zero, we obtain

$$(k_1 + k_2)\,x_1 - k_2\,x_2 = (k_1 - k_2)\,a \tag{17.118}$$

$$-k_2\,x_1 + (k_2 + k_3)\,x_2 = (k_2 - k_3)\,a + k_3 L\ . \tag{17.119}$$

Solving these two inhomogeneous coupled linear equations for $x_{1,2}$ then yields the equilibrium positions. However, we don't need to do this to solve the problem.

(b) Find the T and V matrices.
[5 points]

We have

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot\eta_\sigma\,\partial\dot\eta_{\sigma'}} = \begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix} \tag{17.120}$$

and

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\eta_\sigma\,\partial\eta_{\sigma'}} = \begin{pmatrix} 5k & -4k \\ -4k & 32k \end{pmatrix}\ . \tag{17.121}$$

(c) Find the eigenfrequencies ω_1 and ω_2.
[5 points]

We have

$$\begin{aligned} \mathrm{Q}(\omega) \equiv \omega^2\,\mathrm{T} - \mathrm{V} &= \begin{pmatrix} m\omega^2 - 5k & 4k \\ 4k & 4m\omega^2 - 32k \end{pmatrix} \\ &= k\begin{pmatrix} \lambda - 5 & 4 \\ 4 & 4\lambda - 32 \end{pmatrix}\ , \end{aligned} \tag{17.122}$$

where $\lambda = \omega^2/\omega_0^2$, with $\omega_0 = \sqrt{k/m}$. Setting $\det \mathrm{Q}(\omega) = 0$ then yields

$$\lambda^2 - 13\lambda + 36 = 0 \ , \tag{17.123}$$

the roots of which are $\lambda_- = 4$ and $\lambda_+ = 9$. Thus, the eigenfrequencies are

$$\omega_- = 2\,\omega_0 \qquad , \qquad \omega_+ = 3\,\omega_0 \ . \tag{17.124}$$

(d) Find the modal matrix $\mathrm{A}_{\sigma i}$.
[5 points]

To find the normal modes, we set

$$\begin{pmatrix} \lambda_\pm - 5 & 4 \\ 4 & 4\lambda_\pm - 32 \end{pmatrix} \begin{pmatrix} \psi_1^{(\pm)} \\ \psi_2^{(\pm)} \end{pmatrix} = 0 \ . \tag{17.125}$$

This yields two linearly dependent equations, from which we can determine only the ratios $\psi_2^{(\pm)}/\psi_1^{(\pm)}$. Plugging in for $\lambda_\pm$, we find

$$\begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = \mathcal{C}_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad , \qquad \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ . \tag{17.126}$$

We then normalize by demanding $\psi_\sigma^{(i)}\, \mathrm{T}_{\sigma\sigma'}\, \psi_{\sigma'}^{(j)} = \delta_{ij}$. We can practically solve this by inspection:

$$20m\,|\mathcal{C}_-|^2 = 1 \qquad , \qquad 5m\,|\mathcal{C}_+|^2 = 1 \ . \tag{17.127}$$

We may now write the modal matrix,

$$\mathrm{A} = \frac{1}{\sqrt{5m}} \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \ . \tag{17.128}$$

(e) Write down the most general solution for the motion of the system.
[5 points]

The most general solution is

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = B_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cos(2\omega_0 t + \varphi_-) + B_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(3\omega_0 t + \varphi_+) \ . \tag{17.129}$$

Note that there are four constants of integration: $B_\pm$ and $\varphi_\pm$.

17.7 W08 Physics 110B Midterm Exam

[1] Two identical semi-infinite lengths of string are joined at a point of mass m which moves vertically along a thin wire, as depicted in fig. 17.21. The mass moves with friction coefficient γ, *i.e.* its equation of motion is

$$m\ddot{z} + \gamma\dot{z} = F \ , \tag{17.130}$$

where z is the vertical displacement of the mass, and F is the force on the mass due to the string segments on either side. In this problem, gravity is to be neglected. It may be convenient to define $K \equiv 2\tau/mc^2$ and $Q \equiv \gamma/mc$.

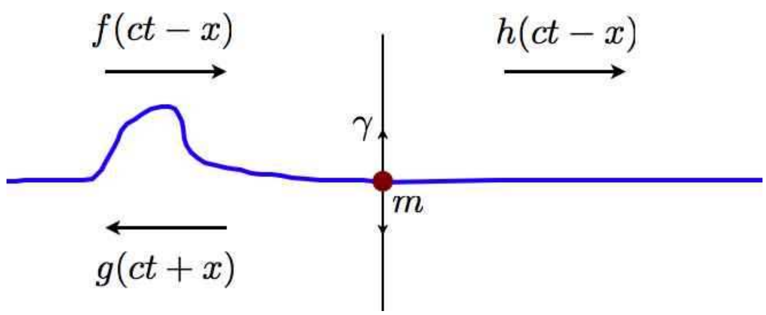

Figure 17.19: A point mass m joining two semi-infinite lengths of identical string moves vertically along a thin wire with friction coefficient γ.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct-x) + g(ct+x) & (x<0) \\ h(ct-x) & (x>0) \ . \end{cases}$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$.
[20 points]

The first equation is continuity at $x=0$:

$$\boxed{f(\xi) = g(\xi) + h(\xi)}$$

where $\xi = ct$ ranges over the real line $[-\infty, \infty]$. The second equation comes from Newton's 2nd law $F = ma$ applied to the mass point:

$$m\,\ddot{y}(0,t) + \gamma\,\dot{y}(0,t) = \tau\,y'(0^+,t) - \tau\,y'(0^-,t) \ .$$

Expressed in terms of the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$, and dividing through by mc^2, this gives

$$f''(\xi) + g''(\xi) + Q\,f'(\xi) + Q\,g'(\xi) = -\tfrac{1}{2}\,K\,h'(\xi) + \tfrac{1}{2}\,K\,f'(\xi) - \tfrac{1}{2}\,K\,g'(\xi).$$

Integrating once, and invoking $h = f + g$, this second equation becomes

$$\boxed{f'(\xi) + Q\, f(\xi) = -g'(\xi) - (K+Q)\, g(\xi)}$$

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \ ,$$

et cetera for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\big|r(k)\big|^2 + \big|t(k)\big|^2$. Show that this sum is always less than or equal to unity, and interpret this fact.
[20 points]

Using $d/d\xi \longrightarrow ik$, we have

$$(Q + ik)\, \hat{f}(k) = -(K + Q + ik)\, \hat{g}(k) \ . \tag{17.131}$$

Therefore,

$$\boxed{r(k) = \frac{\hat{g}(k)}{\hat{f}(k)} = -\frac{Q + ik}{Q + K + ik}} \tag{17.132}$$

To find the transmission amplitude, we invoke $h(\xi) = f(\xi) + g(\xi)$, in which case

$$\boxed{t(k) = \frac{\hat{h}(k)}{\hat{f}(k)} = -\frac{K}{Q + K + ik}} \tag{17.133}$$

The sum of reflection and transmission coefficients is

$$\boxed{\big|r(k)\big|^2 + \big|t(k)\big|^2 = \frac{Q^2 + K^2 + k^2}{(Q+K)^2 + k^2}} \tag{17.134}$$

Clearly the RHS of this equation is bounded from above by unity, since both Q and K are nonnegative.

(c) Find an expression in terms of the functions f, g, and h (and/or their derivatives) for the rate $\dot{E}$ at which energy is lost by the string. Do this by evaluating the energy current on either side of the point mass. Your expression should be an overall function of time t.
[10 points]

Recall the formulae for the energy density in a string,

$$\mathcal{E}(x,t) = \tfrac{1}{2}\, \mu\, \dot{y}^2(x,t) + \tfrac{1}{2}\, \tau\, {y'}^2(x,t) \tag{17.135}$$

and

$$j_{\mathcal{E}}(x,t) = -\tau\,\dot{y}(x,t)\,y'(x,t)\ . \tag{17.136}$$

The energy continuity equation is $\partial_t \mathcal{E} + \partial_x j_{\mathcal{E}} = 0$. Assuming $j_{\mathcal{E}}(\pm\infty,t) = 0$, we have

$$\begin{aligned}\frac{dE}{dt} &= \int\limits_{-\infty}^{0^-}\!\! dx\,\frac{\partial\mathcal{E}}{\partial t} + \int\limits_{0^+}^{\infty}\!\! dx\,\frac{\partial\mathcal{E}}{\partial t}\\ &= -j_{\mathcal{E}}(\infty,t) + j_{\mathcal{E}}(0^+,t) + j_{\mathcal{E}}(-\infty,t) - j_{\mathcal{E}}(0^-,t)\ .\end{aligned} \tag{17.137}$$

Thus,

$$\boxed{\frac{dE}{dt} = c\tau\Big(\big[g'(ct)\big]^2 + \big[h'(ct)\big]^2 - \big[f'(ct)\big]^2\Big)} \tag{17.138}$$

Incidentally, if we integrate over all time, we obtain the total energy change in the string:

$$\begin{aligned}\Delta E &= \tau\!\int\limits_{-\infty}^{\infty}\!\! d\xi\,\Big(\big[g'(\xi)\big]^2 + \big[h'(\xi)\big]^2 - \big[f'(\xi)\big]^2\Big)\\ &= -\tau\!\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,\frac{2QK\,k^2}{(Q+K)^2+k^2}\,\big|\hat{f}(k)\big|^2\ .\end{aligned} \tag{17.139}$$

Note that the initial energy in the string, at time $t = -\infty$, is

$$E_0 = \tau\!\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,k^2\,\big|\hat{f}(k)\big|^2\ . \tag{17.140}$$

If the incident wave packet is very broad, say described by a Gaussian $f(\xi) = A\,\exp(-x^2/2\sigma^2)$ with $\sigma K \gg 1$ and $\sigma Q \gg 1$, then k^2 may be neglected in the denominator of eqn. 17.139, in which case

$$\Delta E \approx -\frac{2QK}{(Q+K)^2}\,E_0 \geq -\tfrac{1}{2}E_0\ . \tag{17.141}$$

[**2**] Consider a rectangular cube of density ρ and dimensions $a \times b \times c$, as depicted in fig. 17.22.

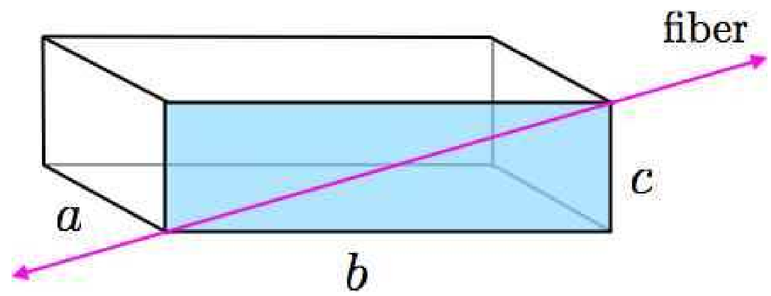

Figure 17.20: A rectangular cube of dimensions $a \times b \times c$. In part (c), a massless torsional fiber is attached along the diagonal of one of the $b \times c$ faces.

(a) Compute the inertia tensor $I_{\alpha\beta}$ along body-fixed principle axes, with the origin at the center of mass.
[15 points]

We first compute I_{zz}:

$$I^{\rm CM}_{zz} = \rho \int\limits_{-a/2}^{a/2} \! dx \int\limits_{-b/2}^{b/2} \! dy \int\limits_{-c/2}^{c/2} \! dz \, \left(x^2 + y^2\right) = \tfrac{1}{12}\, M \left(a^2 + b^2\right) \ , \qquad (17.142)$$

where $M = \rho\, abc$. Corresponding expressions hold for the other moments of inertia. Thus,

$$\boxed{I^{\rm CM} = \tfrac{1}{12} M \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}} \qquad (17.143)$$

(b) Shifting the origin to the center of either of the $b \times c$ faces, and keeping the axes parallel, compute the new inertia tensor.
[15 points]

We shift the origin by a distance $\boldsymbol{d} = -\frac{1}{2} a\, \hat{\boldsymbol{x}}$ and use the parallel axis theorem,

$$I_{\alpha\beta}(\boldsymbol{d}) = I_{\alpha\beta}(0) + M \big(\boldsymbol{d}^2 \delta_{\alpha\beta} - d_\alpha d_\beta \big) \ , \qquad (17.144)$$

resulting in

$$\boxed{I = \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & 4a^2 + c^2 & 0 \\ 0 & 0 & 4a^2 + b^2 \end{pmatrix}} \qquad (17.145)$$

(c) A massless torsional fiber is (masslessly) welded along the diagonal of either $b \times c$ face. The potential energy in this fiber is given by $U(\theta) = \frac{1}{2} Y \theta^2$, where Y is a constant and θ is

the angle of rotation of the fiber. Neglecting gravity, find an expression for the oscillation frequency of the system.
`[20 points]`

Let θ be the twisting angle of the fiber. The kinetic energy in the fiber is

$$\begin{aligned} T &= \tfrac{1}{2}\, I_{\alpha\beta}\, \omega_\alpha\, \omega_\beta \\ &= \tfrac{1}{2}\, n_\alpha\, I_{\alpha\beta}\, n_\beta\, \dot{\theta}^2 \ , \end{aligned} \tag{17.146}$$

where

$$\hat{\boldsymbol{n}} = \frac{b\,\hat{\boldsymbol{y}}}{\sqrt{b^2+c^2}} + \frac{c\,\hat{\boldsymbol{z}}}{\sqrt{b^2+c^2}} \ . \tag{17.147}$$

We then find

$$I_{\rm axis} \equiv n_\alpha\, I_{\alpha\beta}\, n_\beta = \tfrac{1}{3} M a^2 + \tfrac{1}{6} M \frac{b^2\, c^2}{b^2+c^2} \ . \tag{17.148}$$

The frequency of oscillation is then $\Omega = \sqrt{Y/I_{\rm axis}}$, or

$$\boxed{\Omega = \sqrt{\frac{6Y}{M} \cdot \frac{b^2+c^2}{2a^2\big(b^2+c^2\big) + b^2\, c^2}}} \tag{17.149}$$

17.8 W08 Physics 110B Final Exam

[1] Consider a string with uniform mass density μ and tension τ. At the point $x = 0$, the string is connected to a spring of force constant K, as shown in the figure below.

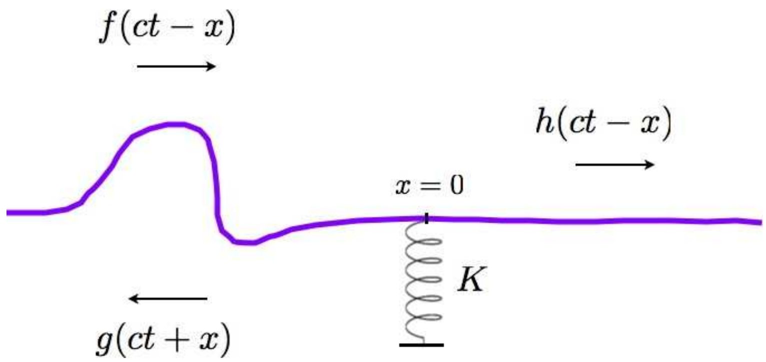

Figure 17.21: A string connected to a spring.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct - x) + g(ct + x) & (x < 0) \\ h(ct - x) & (x > 0) \ . \end{cases}$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$. [10 points]

SOLUTION : The first equation is continuity at $x = 0$:

$$\boxed{f(\xi) + g(\xi) = h(\xi)}$$

where $\xi = ct$ ranges over the real line $[-\infty, \infty]$. The second equation comes from Newton's 2nd law $F = ma$ applied to the mass point:

$$\tau \, y'(0^+, t) - \tau \, y'(0^-, t) - K \, y(0, t) = 0 \ ,$$

or

$$\boxed{-\tau \, h'(\xi) + \tau \, f'(\xi) - \tau \, g'(\xi) - K \big[f(\xi) + g(\xi) \big] = 0}$$

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \, \hat{f}(k) \, e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi \, f(\xi) \, e^{-ik\xi} \ ,$$

et cetera for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\left|r(k)\right|^2 + \left|t(k)\right|^2$. [10 points]

SOLUTION : Taking the Fourier transform of the two equations from part (a), we have

$$\hat{f}(k) + \hat{g}(k) = \hat{h}(k)$$

$$\hat{f}(k) + \hat{g}(k) = \frac{i\tau k}{K}\Big[\hat{f}(k) - \hat{g}(k) - \hat{h}(k)\Big] \ .$$

Solving for $\hat{g}(k)$ and $\hat{h}(k)$ in terms of $\hat{f}(k)$, we find

$$\boxed{\hat{g}(k) = r(k)\,\hat{f}(k) \qquad , \qquad \hat{h}(k) = t(k)\,\hat{f}(k)}$$

where the reflection coefficient $r(k)$ and the transmission coefficient $t(k)$ are given by

$$\boxed{r(k) = -\frac{K}{K + 2i\tau k} \qquad , \qquad t(k) = \frac{2i\tau k}{K + 2i\tau k}}$$

Note that

$$\boxed{\left|r(k)\right|^2 + \left|t(k)\right|^2 = 1}$$

which says that the energy flux is conserved.

(c) For the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 - \tfrac{1}{4}\gamma\left(\frac{\partial y}{\partial x}\right)^4 \ ,$$

find the Euler-Lagrange equations of motion. [7 points]

SOLUTION : For a Lagrangian density $\mathcal{L}(y, \dot{y}, y')$, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) \ .$$

Thus, the wave equation for this system is

$$\boxed{\mu\,\ddot{y} = \tau\,y'' + 3\gamma\left(y'\right)^2 y''}$$

(d) For the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\alpha\, y^2 - \tfrac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2 - \tfrac{1}{4}\beta \left(\frac{\partial^2 y}{\partial x^2}\right)^2 ,$$

find the Euler-Lagrange equations of motion. [7 points]

SOLUTION : For a Lagrangian density $\mathcal{L}(y, \dot{y}, y', y'')$, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial y''}\right) .$$

The last term arises upon integrating by parts twice in the integrand of the variation of the action δS. Thus, the wave equation for this system is

$$\boxed{\mu\, \ddot{y} = -\alpha\, y + \tau\, y'' - \beta\, y''''}$$

[2] Consider single species population dynamics governed by the differential equation

$$\frac{dN}{dt} = \gamma N - \frac{N^2}{K} - \frac{HN}{N+L} \; ,$$

where γ, K, L, and H are constants.

(a) Show that by rescaling N and t that the above ODE is equivalent to

$$\frac{du}{ds} = r\,u - u^2 - \frac{h\,u}{u+1} \; .$$

Give the definitions of u, s, r, and h. [5 points]

SOLUTION : From the denominator $u+1$ in the last term of the scaled equation, we see that we need to define $N = Lu$. We then write $t = \tau s$, and substituting into the original ODE yields

$$\frac{L}{\tau}\,\frac{du}{ds} = \gamma L u - \frac{L^2}{K}\,u^2 - \frac{H\,u}{u+1} \; .$$

Multiplying through by τ/L then gives

$$\frac{du}{ds} = \gamma\tau\,u - \frac{L\tau}{K}\,u^2 - \frac{\tau H}{L}\,\frac{u}{u+1} \; .$$

We set the coefficient of the second term on the RHS equal to -1 to obtain the desired form. Thus, $\tau = K/L$ and

$$\boxed{u = \frac{N}{L} \quad , \quad s = \frac{Lt}{K} \quad , \quad r = \frac{\gamma K}{L} \quad , \quad h = \frac{KH}{L^2}}$$

(b) Find and solve the equation for all fixed points $u^*(r,h)$. [10 points]

SOLUTION : In order for u to be a fixed point, we need $\dot{u} = 0$, which requires

$$\boxed{u\left(r - u - \frac{h}{u+1}\right) = 0}$$

One solution is always $\boxed{u^* = 0}$. The other roots are governed by the quadratic equation

$$(u - r)(u+1) + h = 0 \; ,$$

with roots at

$$\boxed{u^* = \tfrac{1}{2}\left(r - 1 \pm \sqrt{(r+1)^2 - 4h}\right)}$$

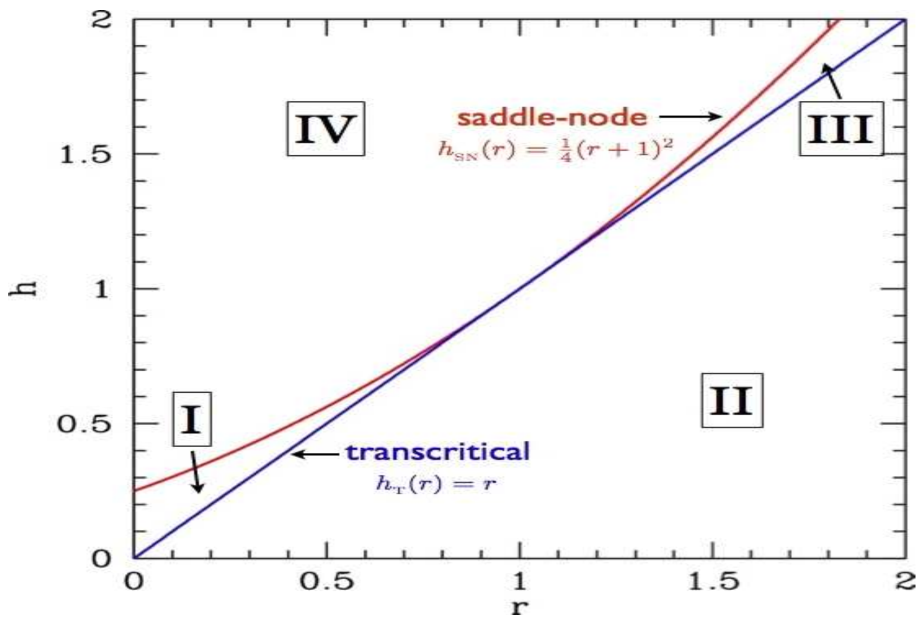

Figure 17.22: Bifurcation curves for the equation $\dot{u} = ru - u^2 - hu/(u+1)$. Red curve: $h_{SN}(r) = \frac{1}{4}(r+1)^2$, corresponding to saddle-node bifurcation. Blue curve: $h_T(r) = r$, corresponding to transcritical bifurcation.

(c) Sketch the upper right quadrant of the (r, h) plane. Show that there are four distinct regions:

Region I	:	3 real fixed points (two negative)
Region II	:	3 real fixed points (one positive, one negative)
Region III	:	3 real fixed points (two positive)
Region IV	:	1 real fixed point

Find the equations for the boundaries of these regions. These boundaries are the locations of bifurcations. Classify the bifurcations. (Note that negative values of u are unphysical in the context of population dynamics, but are legitimate from a purely mathematical standpoint.) [10 points]

SOLUTION : From the quadratic equation for the non-zero roots, we see the discriminant vanishes for $h = \frac{1}{4}(r+1)^2$. For $h > \frac{1}{4}(r+1)^2$, the discriminant is negative, and there is one real root at $u^* = 0$. Thus, the curve $\boxed{h_{SN}(r) = \frac{1}{4}(r+1)^2}$ corresponds to a curve of saddle-node bifurcations. Clearly the largest value of u^* must be a stable node, because for large u the $-u^2$ dominates on the RHS of $\dot{u} = f(u)$. In cases where there are three fixed points, the middle one must be unstable, and the smallest stable. There is another bifurcation, which occurs when the root at $u^* = 0$ is degenerate. This occurs at

$$r - 1 = \sqrt{(r+1)^2 - 4h} \qquad \Longrightarrow \qquad h = r \; .$$

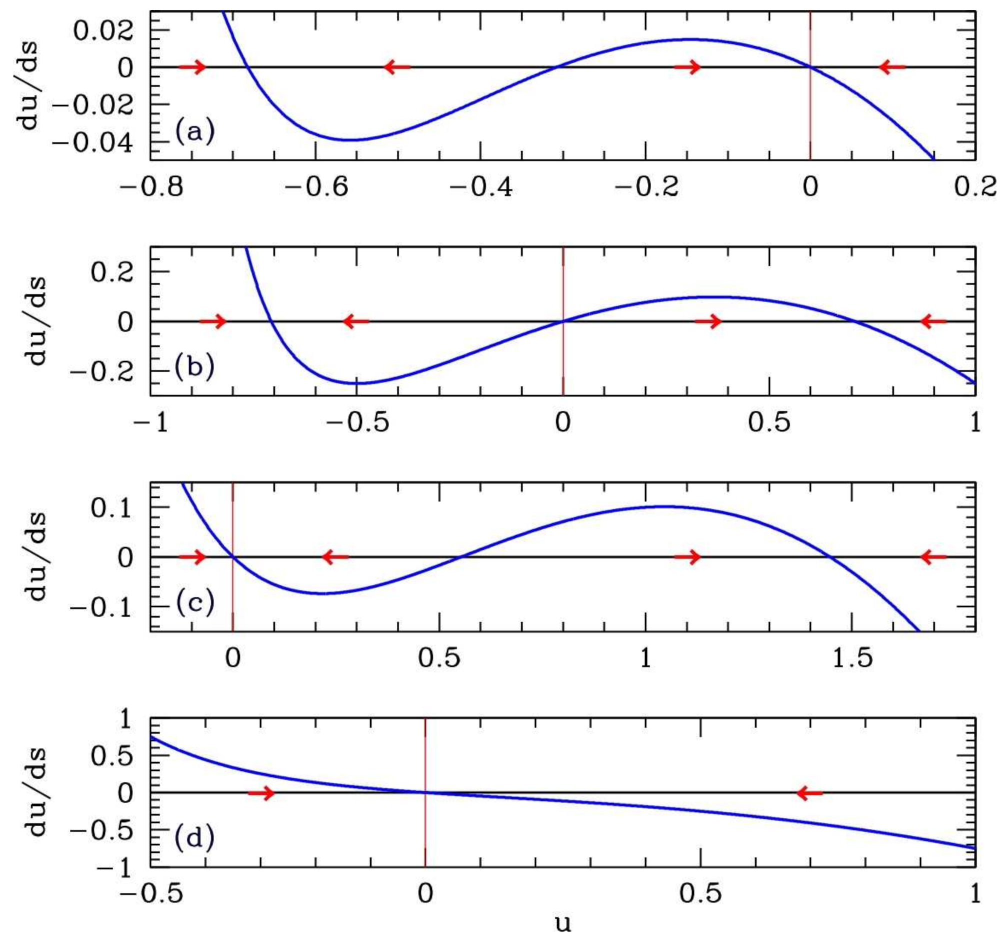

Figure 17.23: Examples of phase flows for the equation $\dot{u} = ru - u^2 - hu/(u+1)$. (a) $r = 1$, $h = 0.22$ (region I) ; (b) $r = 1$, $h = 0.5$ (region II) ; (c) $r = 3$, $h = 3.8$ (region III) ; (d) $r = 1$, $h = 1.5$ (region IV).

This defines the curve for transcritical bifurcations: $\boxed{h_{\rm T}(r) = r}$. Note that $h_{\rm T}(r) \leq h_{\rm SN}(r)$, since $h_{\rm SN}(r) - h_{\rm T}(r) = \frac{1}{4}(r-1)^2 \geq 0$. For $h < r$, one root is positive and one negative, corresponding to region II.

The (r, h) control parameter space is depicted in fig. 17.22, with the regions I through IV bounded by sections of the bifurcation curves, as shown.

(d) Sketch the phase flow for each of the regions I through IV. [8 points]

SOLUTION : See fig. 17.23.

[3] Two brief relativity problems:

(a) A mirror lying in the (x, y) plane moves in the $\hat{\boldsymbol{z}}$ direction with speed u. A monochromatic ray of light making an angle θ with respect to the $\hat{\boldsymbol{z}}$ axis in the laboratory frame reflects off the moving mirror. Find (i) the angle of reflection, measured in the laboratory frame, and (ii) the frequency of the reflected light. [17 points]

SOLUTION : The reflection is simplest to consider in the frame of the mirror, where $\tilde{p}_z \to -\tilde{p}_z$ upon reflection. In the laboratory frame, the 4-momentum of a photon in the beam is

$$P^\mu = \big(E\,,\,0\,,\,E\sin\theta\,,\,E\cos\theta\big)\ ,$$

where, without loss of generality, we have taken the light ray to lie in the (y, z) plane, and where we are taking $c = 1$. Lorentz transforming to the frame of the mirror, we have

$$\tilde{P}^\mu = \big(\gamma E(1 - u\cos\theta)\,,\,0\,,\,E\sin\theta\,,\,\gamma E(-u + \cos\theta)\big)\ .$$

which follows from the general Lorentz boost of a 4-vector Q^μ,

$$\begin{aligned}
\tilde{Q}^0 &= \gamma Q^0 - \gamma u Q_{\parallel} \\
\tilde{Q}_{\parallel} &= -\gamma u Q^0 + \gamma Q_{\parallel} \\
\tilde{\boldsymbol{Q}}_{\perp} &= \boldsymbol{Q}_{\perp}\ ,
\end{aligned}$$

where frame $\tilde{K}$ moves with velocity $\boldsymbol{u}$ with respect to frame K.

Upon reflection, we reverse the sign of $\tilde{P}^3$ in the frame of the mirror:

$$\tilde{P}'^\mu = \big(\gamma E(1 - u\cos\theta)\,,\,0\,,\,E\sin\theta\,,\,\gamma E(u - \cos\theta)\big)\ .$$

Transforming this back to the laboratory frame yields

$$\begin{aligned}
E' = P'^0 &= \gamma^2 E\,(1 - u\cos\theta) + \gamma^2 E\,u\,(u - \cos\theta) \\
&= \gamma^2 E\,\big(1 - 2u\,\cos\theta + u^2\big)
\end{aligned}$$

$$P'^1 = 0$$

$$P'^2 = E\sin\theta$$

$$\begin{aligned}
P'^3 &= \gamma^2 E\,u\,(1 - u\cos\theta) + \gamma^2 E\,(u - \cos\theta) \\
&= -\gamma^2 E\,\big((1 + u^2)\cos\theta - 2u\big)
\end{aligned}$$

Thus, the angle of reflection is

$$\boxed{\cos\theta' = \left|\frac{P'^3}{P'^0}\right| = \frac{(1 + u^2)\cos\theta - 2u}{1 - 2u\cos\theta + u^2}}$$

and the reflected photon frequency is $\nu' = E'/h$, where

$$\boxed{E' = \left(\frac{1 - 2u\,\cos\theta + u^2}{1-u^2}\right) E}$$

(b) Consider the reaction $\pi^+ + \mathrm{n} \to \mathrm{K}^+ + \Lambda^0$. What is the threshold kinetic energy of the pion to create kaon at an angle of 90° in the rest frame of the neutron? Express your answer in terms of the masses m_π, m_n, m_K, and m_Λ. [16 points]

SOLUTION : We have conservation of 4-momentum, giving

$$P^\mu_\pi + P^\mu_\mathrm{n} = P^\mu_\mathrm{K} + P^\mu_\Lambda \ .$$

Thus,

$$\begin{aligned} P^2_\Lambda &= (E_\pi + E_\mathrm{n} - E_\mathrm{K})^2 - (\boldsymbol{P}_\pi + \boldsymbol{P}_\mathrm{n} - \boldsymbol{P}_\mathrm{K})^2 \\ &= (E^2_\pi - \boldsymbol{P}^2_\pi) + (E^2_\mathrm{n} - \boldsymbol{P}^2_\mathrm{n}) + (E^2_\mathrm{K} - \boldsymbol{P}^2_\mathrm{K}) \\ &\qquad + 2E_\pi E_\mathrm{n} - 2E_\pi E_\mathrm{K} - 2E_\mathrm{n} E_\mathrm{K} - 2\boldsymbol{P}_\pi \cdot \boldsymbol{P}_\mathrm{n} + 2\boldsymbol{P}_\pi \cdot \boldsymbol{P}_\mathrm{K} + 2\boldsymbol{P}_\mathrm{n} \cdot \boldsymbol{P}_\mathrm{K} \\ &= E^2_\Lambda - \boldsymbol{P}^2_\Lambda = m^2_\Lambda \ . \end{aligned}$$

Now in the laboratory frame the neutron is at rest, so

$$P^\mu_\mathrm{n} = (m_\mathrm{n}\,,\,\mathbf{0}) \ .$$

Thus, $\mathbf{P}_\pi \cdot \mathbf{P}_\mathrm{n} = \mathbf{P}_\mathrm{n} \cdot \mathbf{P}_\mathrm{K} = 0$. We are also told that the pion and the kaon make an angle of 90° in the laboratory frame, hence $\mathbf{P}_\pi \cdot \mathbf{P}_\mathrm{K} = 0$. And of course for each particle we have $E^2 - \mathbf{P}^2 = m^2$. Thus, we have

$$m^2_\Lambda = m^2_\pi + m^2_\mathrm{n} + m^2_\mathrm{K} - 2m_\mathrm{n} E_\mathrm{K} + 2(m_\mathrm{n} - E_\mathrm{K})\, E_\pi \ ,$$

or, solving for E_π,

$$E_\pi = \frac{m^2_\Lambda - m^2_\pi - m^2_\mathrm{n} - m^2_\mathrm{K} + 2m_\mathrm{n} E_\mathrm{K}}{2(m_\mathrm{n} - E_\mathrm{K})} \ .$$

The threshold pion energy is the minimum value of E_π, which must occur when E_K takes its minimum allowed value, $E_\mathrm{K} = m_\mathrm{K}$. Thus,

$$\boxed{T_\pi = E_\pi - m_\pi \geq \frac{m^2_\Lambda - m^2_\pi - m^2_\mathrm{n} - m^2_\mathrm{K} + 2m_\mathrm{n} m_\mathrm{K}}{2(m_\mathrm{n} - m_\mathrm{K})} - m_\pi}$$

[4] Sketch what a bletch might look like. [10,000 quatloos extra credit]
[-50 points if it looks like your professor]

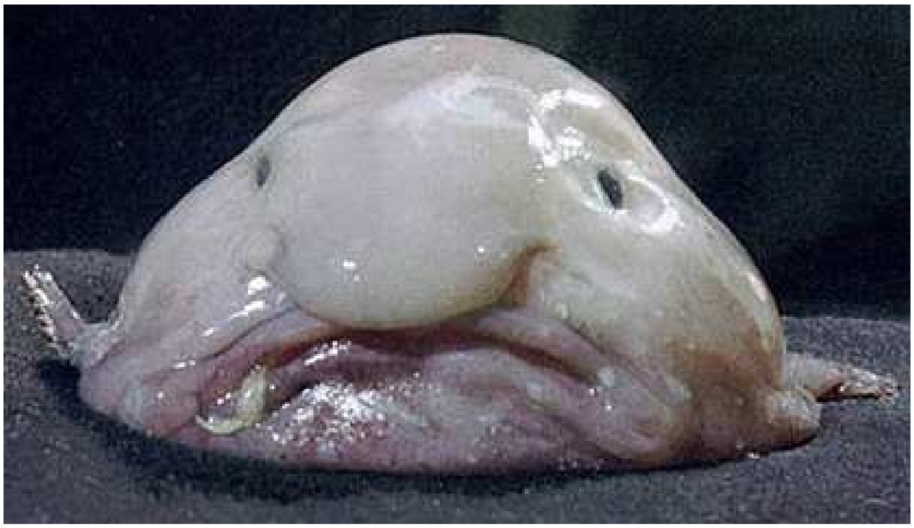

Figure 17.24: The putrid bletch, from the (underwater) Jkroo forest, on planet Barney.