Chapter 8

Constraints

A mechanical system of N point particles in d dimensions possesses $n = dN$ degrees of freedom[1]. To specify these degrees of freedom, we can choose any independent set of generalized coordinates $\{q_1, \ldots, q_K\}$. Oftentimes, however, not all n coordinates are independent.

Consider, for example, the situation in Fig. 8.1, where a cylinder of radius a rolls over a half-cylinder of radius R. If there is no slippage, then the angles θ_1 and θ_2 are not independent, and they obey the *equation of constraint*,

$$R\,\theta_1 = a\,(\theta_2 - \theta_1)\ . \tag{8.1}$$

In this case, we can easily solve the constraint equation and substitute $\theta_2 = \big(1 + \frac{R}{a}\big)\,\theta_1$. In other cases, though, the equation of constraint might not be so easily solved (*e.g.* it may be nonlinear). How then do we proceed?

8.1 Constraints and Variational Calculus

Before addressing the subject of constrained dynamical systems, let's consider the issue of constraints in the broader context of variational calculus. Suppose we have a functional

$$F\big[y(x)\big] = \int\limits_{x_a}^{x_b} \!dx\, L(y, y', x)\ , \tag{8.2}$$

which we want to extremize subject to some constraints. Here y may stand for a set of functions $\{y_\sigma(x)\}$. There are two classes of constraints we will consider:

[1] For N rigid bodies, the number of degrees of freedom is $n' = \frac{1}{2}d(d+1)N$, corresponding to d center-of-mass coordinates and $\frac{1}{2}d(d-1)$ angles of orientation for each particle. The dimension of the group of rotations in d dimensions is $\frac{1}{2}d(d-1)$, corresponding to the number of parameters in a general rank-d orthogonal matrix (*i.e.* an element of the group $O(d)$).

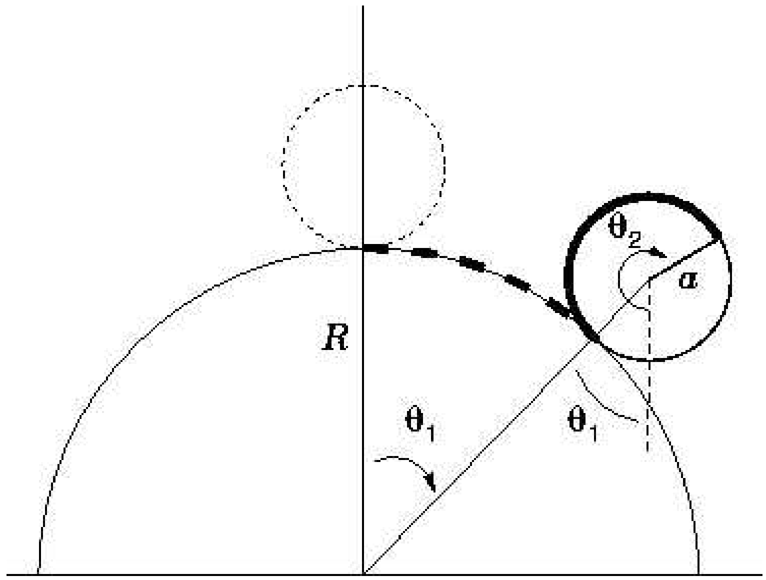

Figure 8.1: A cylinder of radius a rolls along a half-cylinder of radius R. When there is no slippage, the angles θ_1 and θ_2 obey the constraint equation $R\theta_1 = a(\theta_2 - \theta_1)$.

1. *Integral constraints:* These are of the form

$$\int\limits_{x_a}^{x_b} dx\, N_j(y, y', x) = C_j \ , \tag{8.3}$$

where j labels the constraint.

2. *Holonomic constraints:* These are of the form

$$G_j(y, x) = 0 \ . \tag{8.4}$$

The cylinders system in Fig. 8.1 provides an example of a holonomic constraint. There, $G(\theta, t) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0$. As an example of a problem with an integral constraint, suppose we want to know the shape of a hanging rope of fixed length C. This means we minimize the rope's potential energy,

$$U[y(x)] = \lambda g \int\limits_{x_a}^{x_b} ds\, y(x) = \lambda g \int\limits_{x_a}^{x_b} dx\, y \sqrt{1 + y'^2} \ , \tag{8.5}$$

where λ is the linear mass density of the rope, subject to the fixed-length constraint

$$C = \int\limits_{x_a}^{x_b} ds = \int\limits_{x_a}^{x_b} dx\, \sqrt{1 + y'^2} \ . \tag{8.6}$$

Note $ds = \sqrt{dx^2 + dy^2}$ is the differential element of arc length along the rope. To solve problems like these, we turn to Lagrange's method of *undetermined multipliers*.

8.2 Constrained Extremization of Functions

Given $F(x_1, \ldots, x_n)$ to be extremized subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ where $j = 1, \ldots, k$, construct

$$F^*\big(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \, G_j(x_1, \ldots, x_n) \tag{8.7}$$

which is a function of the $(n+k)$ variables $\big\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big\}$. Now freely extremize the extended function F^*:

$$dF^* = \sum_{\sigma=1}^{n} \frac{\partial F^*}{\partial x_\sigma} \, dx_\sigma + \sum_{j=1}^{k} \frac{\partial F^*}{\partial \lambda_j} \, d\lambda_j \tag{8.8}$$

$$= \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j = 0 \tag{8.9}$$

This results in the $(n+k)$ equations

$$\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, n) \tag{8.10}$$

$$G_j = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.11}$$

The interpretation of all this is as follows. The n equations in 8.10 can be written in vector form as

$$\boldsymbol{\nabla} F + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{\nabla} G_j = 0 \ . \tag{8.12}$$

This says that the (n-component) vector $\boldsymbol{\nabla} F$ is linearly dependent upon the k vectors $\boldsymbol{\nabla} G_j$. Thus, any movement in the direction of $\boldsymbol{\nabla} F$ must necessarily entail movement along one or more of the directions $\boldsymbol{\nabla} G_j$. This would require violating the constraints, since movement along $\boldsymbol{\nabla} G_j$ takes us off the level set $G_j = 0$. Were $\boldsymbol{\nabla} F$ linearly *independent* of the set $\{\boldsymbol{\nabla} G_j\}$, this would mean that we could find a differential displacement $d\boldsymbol{x}$ which has finite overlap with $\boldsymbol{\nabla} F$ but zero overlap with each $\boldsymbol{\nabla} G_j$. Thus $\boldsymbol{x} + d\boldsymbol{x}$ would still satisfy $G_j(\boldsymbol{x} + d\boldsymbol{x}) = 0$, but F would change by the finite amount $dF = \boldsymbol{\nabla} F(\boldsymbol{x}) \cdot d\boldsymbol{x}$.

8.3 Extremization of Functionals : Integral Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int_{x_a}^{x_b} \! dx \, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{8.13}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$\int\limits_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l = 1,\ldots,k) \ , \tag{8.14}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j\}\big] \equiv \int\limits_{x_a}^{x_b} dx \bigg\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{l=1}^{k} \lambda_l\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big)\bigg\} - \sum_{l=1}^{k} \lambda_l\, C_l \tag{8.15}$$

and freely extremize over $\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\}$. This results in $(n+k)$ equations

$$\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{l=1}^{k} \lambda_l \bigg\{ \frac{\partial N_l}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial N_l}{\partial y'_\sigma}\bigg)\bigg\} = 0 \qquad (\sigma = 1,\ldots,n) \tag{8.16}$$

$$\int\limits_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l = 1,\ldots,k) \ . \tag{8.17}$$

8.4 Extremization of Functionals : Holonomic Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int\limits_{x_a}^{x_b} dx\, L\big(\{y_\sigma\},\{y'_\sigma\},x\big) \qquad (\sigma = 1,\ldots,n) \tag{8.18}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$G_j\big(\{y_\sigma(x)\},x\big) = 0 \qquad (j = 1,\ldots,k) \ , \tag{8.19}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j(x)\}\big] \equiv \int\limits_{x_a}^{x_b} dx \bigg\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{j=1}^{k} \lambda_j\, G_j\big(\{y_\sigma\}\big)\bigg\} \tag{8.20}$$

and freely extremize over $\big\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\big\}$:

$$\delta F^* = \int\limits_{x_a}^{x_b} dx \bigg\{ \sum_{\sigma=1}^{n} \bigg(\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma}\bigg)\delta y_\sigma + \sum_{j=1}^{k} G_j\, \delta\lambda_j \bigg\} = 0 \ , \tag{8.21}$$

resulting in the $(n+k)$ equations

$$\frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma} \qquad (\sigma = 1,\ldots,n) \tag{8.22}$$

$$G_j\big(\{y_\sigma\},x\big) = 0 \qquad (j = 1,\ldots,k) \ . \tag{8.23}$$

8.4.1 Examples of extremization with constraints

Volume of a cylinder : As a warm-up problem, let's maximize the volume $V = \pi a^2 h$ of a cylinder of radius a and height h, subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0\ . \tag{8.24}$$

We therefore define

$$V^*(a,h,\lambda) \equiv V(a,h) + \lambda\, G(a,h)\ , \tag{8.25}$$

and set

$$\frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0 \tag{8.26}$$

$$\frac{\partial V^*}{\partial h} = \pi a^2 + 2\lambda\, \frac{h}{b} = 0 \tag{8.27}$$

$$\frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0\ . \tag{8.28}$$

Solving these three equations simultaneously gives

$$a = \frac{2\ell}{5\pi} \quad , \quad h = \sqrt{\frac{b\ell}{5}} \quad , \quad \lambda = \frac{2\pi}{5^{3/2}}\, b^{1/2}\, \ell^{3/2} \quad , \quad V = \frac{4}{5^{5/2}\,\pi}\, \ell^{5/2}\, b^{1/2}\ . \tag{8.29}$$

Hanging rope : We minimize the energy functional

$$E\big[y(x)\big] = \mu g \int\limits_{x_1}^{x_2}\!dx\, y \sqrt{1+y'^2}\ , \tag{8.30}$$

where μ is the linear mass density, subject to the constraint of fixed total length,

$$C\big[y(x)\big] = \int\limits_{x_1}^{x_2}\!dx\, \sqrt{1+y'^2}\ . \tag{8.31}$$

Thus,

$$E^*\big[y(x),\lambda\big] = E\big[y(x)\big] + \lambda C\big[y(x)\big] = \int\limits_{x_1}^{x_2}\!dx\, L^*(y,y',x)\ , \tag{8.32}$$

with

$$L^*(y,y',x) = (\mu g y + \lambda)\, \sqrt{1+y'^2}\ . \tag{8.33}$$

Since $\frac{\partial L^*}{\partial x} = 0$ we have that

$$\mathcal{J} = y'\,\frac{\partial L^*}{\partial y'} - L^* = -\frac{\mu g y + \lambda}{\sqrt{1+y'^2}} \tag{8.34}$$

is constant. Thus,

$$\frac{dy}{dx} = \pm \mathcal{J}^{-1} \sqrt{(\mu g y + \lambda)^2 - \mathcal{J}^2} \ , \tag{8.35}$$

with solution

$$y(x) = -\frac{\lambda}{\mu g} + \frac{\mathcal{J}}{\mu g} \cosh\left(\frac{\mu g}{\mathcal{J}}\,(x-a)\right) . \tag{8.36}$$

Here, $\mathcal{J}$, a, and λ are constants to be determined by demanding $y(x_i) = y_i$ $(i = 1, 2)$, and that the total length of the rope is C.

Geodesic on a curved surface : Consider next the problem of a geodesic on a curved surface. Let the equation for the surface be

$$G(x, y, z) = 0 \ . \tag{8.37}$$

We wish to extremize the distance,

$$D = \int\limits_a^b ds = \int\limits_a^b \sqrt{dx^2 + dy^2 + dz^2} \ . \tag{8.38}$$

We introduce a parameter t defined on the unit interval: $t \in [0, 1]$, such that $x(0) = x_a$, $x(1) = x_b$, *etc.* Then D may be regarded as a functional, *viz.*

$$D\big[x(t), y(t), z(t)\big] = \int\limits_0^1 dt\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \ . \tag{8.39}$$

We impose the constraint by forming the extended functional, D^*:

$$D^*\big[x(t), y(t), z(t), \lambda(t)\big] \equiv \int\limits_0^1 dt\,\left\{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda\, G(x, y, z)\right\} , \tag{8.40}$$

and we demand that the first functional derivatives of D^* vanish:

$$\frac{\delta D^*}{\delta x(t)} = -\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial x} = 0 \tag{8.41}$$

$$\frac{\delta D^*}{\delta y(t)} = -\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial y} = 0 \tag{8.42}$$

$$\frac{\delta D^*}{\delta z(t)} = -\frac{d}{dt}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial z} = 0 \tag{8.43}$$

$$\frac{\delta D^*}{\delta \lambda(t)} = G(x, y, z) = 0 \ . \tag{8.44}$$

Thus,

$$\lambda(t) = \frac{v\ddot{x} - \dot{x}\dot{v}}{v^2\,\partial_x G} = \frac{v\ddot{y} - \dot{y}\dot{v}}{v^2\,\partial_y G} = \frac{v\ddot{z} - \dot{z}\dot{v}}{v^2\,\partial_z G} \ , \tag{8.45}$$

with $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and $\partial_x \equiv \frac{\partial}{\partial x}$, *etc.* These three equations are supplemented by $G(x, y, z) = 0$, which is the fourth.

8.5 Application to Mechanics

Let us write our system of constraints in the differential form

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, dq_\sigma + h_j(q,t) dt = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.46}$$

If the partial derivatives satisfy

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma} \ , \tag{8.47}$$

then the differential can be integrated to give $dG(q,t) = 0$, where

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad , \quad h_j = \frac{\partial G_j}{\partial t} \ . \tag{8.48}$$

The action functional is

$$S[\{q_\sigma(t)\}] = \int_{t_a}^{t_b} dt\, L\big(\{q_\sigma\}, \{\dot q_\sigma\}, t\big) \qquad (\sigma = 1, \ldots, n) \ , \tag{8.49}$$

subject to boundary conditions $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. The first variation of S is given by

$$\delta S = \int_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) \right\} \delta q_\sigma \ . \tag{8.50}$$

Since the $\{q_\sigma(t)\}$ are no longer independent, we cannot infer that the term in brackets vanishes for each σ. What are the constraints on the variations $\delta q_\sigma(t)$? The constraints are expressed in terms of *virtual displacements* which take no time: $\delta t = 0$. Thus,

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \delta q_\sigma(t) = 0 \ , \tag{8.51}$$

where $j = 1, \ldots, k$ is the constraint index. We may now relax the constraint by introducing k undetermined functions $\lambda_j(t)$, by adding integrals of the above equations with undetermined coefficient functions to δS:

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0 \ . \tag{8.52}$$

Now we can demand that the term in brackets vanish for all σ. Thus, we obtain a set of $(n+k)$ equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \equiv Q_\sigma \tag{8.53}$$

$$g_{j\sigma}(q,t)\, \dot q_\sigma + h_j(q,t) = 0 \ , \tag{8.54}$$

in $(n+k)$ unknowns $\{q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k\}$. Here, Q_σ is the *force of constraint conjugate to the generalized coordinate* q_σ. Thus, with

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma} \quad , \quad F_\sigma = \frac{\partial L}{\partial q_\sigma} \quad , \quad Q_\sigma = \sum_{j=1}^k \lambda_j \, g_{j\sigma} \ , \tag{8.55}$$

we write Newton's second law as

$$\dot p_\sigma = F_\sigma + Q_\sigma \ . \tag{8.56}$$

Note that we can write

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\bigg) \tag{8.57}$$

and that the *instantaneous* constraints may be written

$$\boldsymbol{g}_j \cdot \delta \boldsymbol{q} = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.58}$$

Thus, by demanding

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} + \sum_{j=1}^k \lambda_j \, \boldsymbol{g}_j = 0 \tag{8.59}$$

we require that the functional derivative be linearly dependent on the k vectors $\boldsymbol{g}_j$.

8.5.1 Constraints and conservation laws

We have seen how invariance of the Lagrangian with respect to a one-parameter family of coordinate transformations results in an associated conserved quantity Λ, and how a lack of explicit time dependence in L results in the conservation of the Hamiltonian H. In deriving both these results, however, we used the equations of motion $\dot p_\sigma = F_\sigma$. What happens when we have constraints, in which case $\dot p_\sigma = F_\sigma + Q_\sigma$?

Let's begin with the Hamiltonian. We have $H = \dot q_\sigma \, p_\sigma - L$, hence

$$\begin{aligned} \frac{dH}{dt} &= \bigg(p_\sigma - \frac{\partial L}{\partial \dot q_\sigma}\bigg)\ddot q_\sigma + \bigg(\dot p_\sigma - \frac{\partial L}{\partial q_\sigma}\bigg)\dot q_\sigma - \frac{\partial L}{\partial t} \\ &= Q_\sigma \, \dot q_\sigma - \frac{\partial L}{\partial t} \ . \end{aligned} \tag{8.60}$$

We now use

$$Q_\sigma \, \dot q_\sigma = \lambda_j \, g_{j\sigma} \, \dot q_\sigma = -\lambda_j \, h_j \tag{8.61}$$

to obtain

$$\frac{dH}{dt} = -\lambda_j \, h_j - \frac{\partial L}{\partial t} \ . \tag{8.62}$$

We therefore conclude that *in a system with constraints of the form* $g_{j\sigma} \, \dot q_\sigma + h_j = 0$, *the Hamiltonian is conserved if each* $h_j = 0$ *and if* L *is not explicitly dependent on time.* In the case of holonomic constraints, $h_j = \frac{\partial G_j}{\partial t}$, so H is conserved if neither L nor any of the constraints G_j is explicitly time-dependent.

Next, let us rederive Noether's theorem when constraints are present. We assume a one-parameter family of transformations $q_\sigma \to \tilde{q}_\sigma(\zeta)$ leaves L invariant. Then

$$\begin{aligned} 0 = \frac{dL}{d\zeta} &= \frac{\partial L}{\partial \tilde{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \frac{\partial L}{\partial \dot{\tilde{q}}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta} \\ &= \big(\dot{\tilde{p}}_\sigma - \tilde{Q}_\sigma\big)\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \tilde{p}_\sigma\,\frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) \\ &= \frac{d}{dt}\bigg(\tilde{p}_\sigma\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) - \lambda_j\,\tilde{g}_{j\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\ . \end{aligned} \tag{8.63}$$

Now let us write the constraints in differential form as

$$\tilde{g}_{j\sigma}\,d\tilde{q}_\sigma + \tilde{h}_j\,dt + \tilde{k}_j\,d\zeta = 0\ . \tag{8.64}$$

We now have

$$\frac{d\Lambda}{dt} = \lambda_j\,\tilde{k}_j\ , \tag{8.65}$$

which says that *if the constraints are independent of ζ then Λ is conserved.* For holonomic constraints, this means that

$$G_j\big(\tilde{q}(\zeta),t\big) = 0 \quad \Rightarrow \quad \tilde{k}_j = \frac{\partial G_j}{\partial \zeta} = 0\ , \tag{8.66}$$

i.e. $G_j(\tilde{q},t)$ has no explicit ζ dependence.

8.6 Worked Examples

Here we consider several example problems of constrained dynamics, and work each out in full detail.

8.6.1 One cylinder rolling off another

As an example of the constraint formalism, consider the system in Fig. 8.1, where a cylinder of radius a rolls atop a cylinder of radius R. We have two constraints:

$$G_1(r,\theta_1,\theta_2) = r - R - a = 0 \qquad \text{(cylinders in contact)} \tag{8.67}$$

$$G_2(r,\theta_1,\theta_2) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0 \qquad \text{(no slipping)}\ , \tag{8.68}$$

from which we obtain the $g_{j\sigma}$:

$$g_{j\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix}\ , \tag{8.69}$$

which is to say

$$\frac{\partial G_1}{\partial r} = 1 \qquad \frac{\partial G_1}{\partial \theta_1} = 0 \qquad \frac{\partial G_1}{\partial \theta_2} = 0 \tag{8.70}$$

$$\frac{\partial G_2}{\partial r} = 0 \qquad \frac{\partial G_2}{\partial \theta_1} = R + a \qquad \frac{\partial G_2}{\partial \theta_2} = -a\ . \tag{8.71}$$

The Lagrangian is

$$L = T - U = \tfrac{1}{2} M \big(\dot r^2 + r^2\, \dot\theta_1^2\big) + \tfrac{1}{2} I\, \dot\theta_2^2 - Mgr\, \cos\theta_1\ , \tag{8.72}$$

where M and I are the mass and rotational inertia of the rolling cylinder, respectively. Note that the kinetic energy is a sum of center-of-mass translation $T_{\rm tr} = \frac{1}{2} M \big(\dot r^2 + r^2\, \dot\theta_1^2\big)$ and rotation about the center-of-mass, $T_{\rm rot} = \frac{1}{2} I\, \dot\theta_2^2$. The equations of motion are

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot r}\bigg) - \frac{\partial L}{\partial r} = M\ddot r - Mr\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \equiv Q_r \tag{8.73}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot \theta_1}\bigg) - \frac{\partial L}{\partial \theta_1} = Mr^2 \ddot\theta_1 + 2Mr\dot r\,\dot\theta_1 - Mgr\sin\theta_1 = (R+a)\,\lambda_2 \equiv Q_{\theta_1} \tag{8.74}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot \theta_2}\bigg) - \frac{\partial L}{\partial \theta_2} = I\ddot\theta_2 = -a\,\lambda_2 \equiv Q_{\theta_2}\ . \tag{8.75}$$

To these three equations we add the two constraints, resulting in five equations in the five unknowns $\big\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\big\}$.

We solve by first implementing the constraints, which give $r = (R + a)$ a constant (*i.e.* $\dot r = 0$), and $\dot\theta_2 = \big(1 + \frac{R}{a}\big)\,\dot\theta_1$. Substituting these into the above equations gives

$$-M(R+a)\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \tag{8.76}$$

$$M(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = (R+a)\,\lambda_2 \tag{8.77}$$

$$I\bigg(\frac{R+a}{a}\bigg)\ddot\theta_1 = -a\lambda_2\ . \tag{8.78}$$

From eqn. 8.78 we obtain

$$\lambda_2 = -\frac{I}{a}\,\ddot\theta_2 = -\frac{R+a}{a^2}\,I\,\ddot\theta_1\ , \tag{8.79}$$

which we substitute into eqn. 8.77 to obtain

$$\bigg(M + \frac{I}{a^2}\bigg)(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = 0\ . \tag{8.80}$$

Multiplying by $\dot\theta_1$, we obtain an exact differential, which may be integrated to yield

$$\tfrac{1}{2} M\bigg(1 + \frac{I}{Ma^2}\bigg)\dot\theta_1^2 + \frac{Mg}{R+a}\,\cos\theta_1 = \frac{Mg}{R+a}\cos\theta_1^\circ\ . \tag{8.81}$$

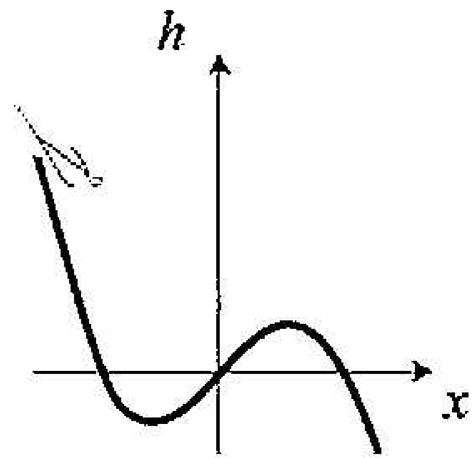

Figure 8.2: Frictionless motion under gravity along a curved surface. The skier flies off the surface when the normal force vanishes.

Here, we have assumed that $\dot{\theta}_1 = 0$ when $\theta_1 = \theta_1^\circ$, *i.e.* the rolling cylinder is released from rest at $\theta_1 = \theta_1^\circ$. Finally, inserting this result into eqn. 8.76, we obtain the radial force of constraint,

$$Q_r = \frac{Mg}{1+\alpha}\Big\{(3+\alpha)\,\cos\theta_1 - 2\,\cos\theta_1^\circ\Big\}\ , \tag{8.82}$$

where $\alpha = I/Ma^2$ is a dimensionless parameter $(0 \le \alpha \le 1)$. This is the radial component of the normal force between the two cylinders. When Q_r vanishes, the cylinders lose contact – the rolling cylinder flies off. Clearly this occurs at an angle $\theta_1 = \theta_1^*$, where

$$\theta_1^* = \cos^{-1}\left(\frac{2\cos\theta_1^\circ}{3+\alpha}\right) . \tag{8.83}$$

The detachment angle θ_1^* is an increasing function of α, which means that larger I delays detachment. This makes good sense, since when I is larger the gain in kinetic energy is split between translational and rotational motion of the rolling cylinder.

8.6.2 Frictionless motion along a curve

Consider the situation in Fig. 8.2 where a skier moves frictionlessly under the influence of gravity along a general curve $y = h(x)$. The Lagrangian for this problem is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{8.84}$$

and the (holonomic) constraint is

$$G(x,y) = y - h(x) = 0\ . \tag{8.85}$$

Accordingly, the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \lambda\,\frac{\partial G}{\partial q_\sigma}\ , \tag{8.86}$$

where $q_1 = x$ and $q_2 = y$. Thus, we obtain

$$m\ddot{x} = -\lambda\, h'(x) = Q_x \tag{8.87}$$

$$m\ddot{y} + mg = \lambda = Q_y \ . \tag{8.88}$$

We eliminate y in favor of x by invoking the constraint. Since we need $\ddot{y}$, we must differentiate the constraint, which gives

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.89}$$

Using the second Euler-Lagrange equation, we then obtain

$$\frac{\lambda}{m} = g + h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.90}$$

Finally, we substitute this into the first E-L equation to obtain an equation for x alone:

$$\Big(1 + \big[h'(x)\big]^2\Big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 + g\,h'(x) = 0 \ . \tag{8.91}$$

Had we started by eliminating $y = h(x)$ at the outset, writing

$$L(x, \dot{x}) = \tfrac{1}{2}m\Big(1 + \big[h'(x)\big]^2\Big)\,\dot{x}^2 - mg\,h(x) \ , \tag{8.92}$$

we would also have obtained this equation of motion.

The skier flies off the curve when the vertical force of constraint $Q_y = \lambda$ starts to become negative, because the curve can only supply a positive normal force. Suppose the skier starts from rest at a height y_0. We may then determine the point x at which the skier detaches from the curve by setting $\lambda(x) = 0$. To do so, we must eliminate $\dot{x}$ and $\ddot{x}$ in terms of x. For $\ddot{x}$, we may use the equation of motion to write

$$\ddot{x} = -\bigg(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\bigg) \ , \tag{8.93}$$

which allows us to write

$$\lambda = m\bigg(\frac{g + h''\,\dot{x}^2}{1 + h'^2}\bigg) \ . \tag{8.94}$$

To eliminate $\dot{x}$, we use conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot{x}^2 + mgh \ , \tag{8.95}$$

which fixes

$$\dot{x}^2 = 2g\bigg(\frac{y_0 - h}{1 + h'^2}\bigg) \ . \tag{8.96}$$

Putting it all together, we have

$$\lambda(x) = \frac{mg}{\big(1 + h'^2\big)^2}\,\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\} \ . \tag{8.97}$$

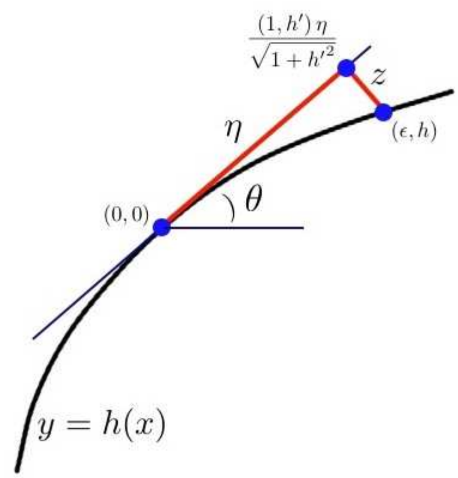

Figure 8.3: Finding the local radius of curvature: $z = \eta^2/2R$.

The skier detaches from the curve when $\lambda(x) = 0$, *i.e.* when

$$1 + h'^2 + 2(y_0 - h)\, h'' = 0 \ . \tag{8.98}$$

There is a somewhat easier way of arriving at the same answer. This is to note that the skier must fly off when the local centripetal force equals the gravitational force normal to the curve, *i.e.*

$$\frac{m\, v^2(x)}{R(x)} = mg\, \cos\theta(x) \ , \tag{8.99}$$

where $R(x)$ is the local radius of curvature. Now $\tan\theta = h'$, so $\cos\theta = \big(1 + h'^2\big)^{-1/2}$. The square of the velocity is $v^2 = \dot{x}^2 + \dot{y}^2 = \big(1 + h'^2\big)\, \dot{x}^2$. What is the local radius of curvature $R(x)$? This can be determined from the following argument, and from the sketch in Fig. 8.3. Writing $x = x^* + \epsilon$, we have

$$y = h(x^*) + h'(x^*)\, \epsilon + \tfrac{1}{2} h''(x^*)\, \epsilon^2 + \ldots \ . \tag{8.100}$$

We now drop a perpendicular segment of length z from the point (x, y) to the line which is tangent to the curve at $\big(x^*, h(x^*)\big)$. According to Fig. 8.3, this means

$$\begin{pmatrix} \epsilon \\ y \end{pmatrix} = \eta \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} 1 \\ h' \end{pmatrix} - z \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} -h' \\ 1 \end{pmatrix} \ . \tag{8.101}$$

Thus, we have

$$\begin{aligned} y &= h'\,\epsilon + \tfrac{1}{2} h''\,\epsilon^2 \\ &= h' \left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}} \right) + \tfrac{1}{2} h'' \left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}} \right)^2 \\ &= \frac{\eta\,h' + z\,h'^2}{\sqrt{1+h'^2}} + \frac{h''\,\eta^2}{2\big(1+h'^2\big)} + \mathcal{O}(\eta z) \\ &= \frac{\eta\,h' - z}{\sqrt{1+h'^2}} \ , \end{aligned} \tag{8.102}$$

from which we obtain

$$z = -\frac{h''\,\eta^2}{2\big(1+h'^2\big)^{3/2}} + \mathcal{O}(\eta^3) \tag{8.103}$$

and therefore

$$R(x) = -\frac{1}{h''(x)} \cdot \Big(1 + \big[h'(x)\big]^2 \Big)^{3/2} \ . \tag{8.104}$$

Thus, the detachment condition,

$$\frac{mv^2}{R} = -\frac{m\,h''\,\dot{x}^2}{\sqrt{1+h'^2}} = \frac{mg}{\sqrt{1+h'^2}} = mg\,\cos\theta \tag{8.105}$$

reproduces the result from eqn. 8.94.

8.6.3 Disk rolling down an inclined plane

A hoop of mass m and radius R rolls without slipping down an inclined plane. The inclined plane has opening angle α and mass M, and itself slides frictionlessly along a horizontal surface. Find the motion of the system.

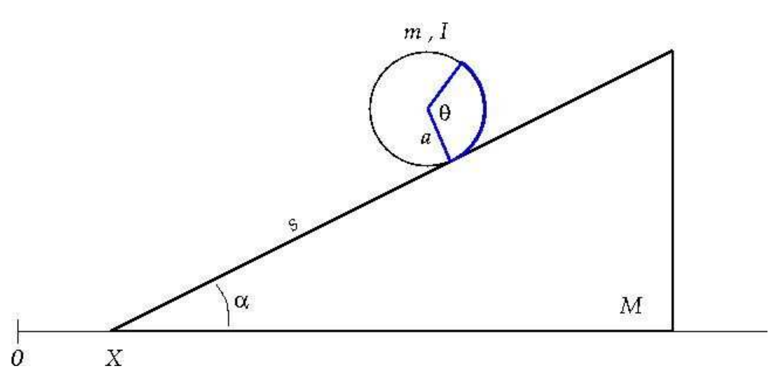

Figure 8.4: A hoop rolling down an inclined plane lying on a frictionless surface.

Solution : Referring to the sketch in Fig. 8.4, the center of the hoop is located at

$$\begin{aligned} x &= X + s\,\cos\alpha - a\,\sin\alpha \\ y &= s\,\sin\alpha + a\,\cos\alpha\ , \end{aligned}$$

where X is the location of the lower left corner of the wedge, and s is the distance along the wedge to the bottom of the hoop. If the hoop rotates through an angle θ, the no-slip condition is $a\,\dot\theta + \dot s = 0$. Thus,

$$\begin{aligned} L &= \tfrac{1}{2} M\dot X^2 + \tfrac{1}{2} m\big(\dot x^2 + \dot y^2\big) + \tfrac{1}{2} I\dot\theta^2 - mgy \\ &= \tfrac{1}{2}\bigg(m + \frac{I}{a^2}\bigg)\dot s^2 + \tfrac{1}{2}(M+m)\dot X^2 + m\cos\alpha\,\dot X\,\dot s - mgs\,\sin\alpha - mga\,\cos\alpha\ . \end{aligned}$$

Since X is cyclic in L, the momentum

$$P_X = (M+m)\dot X + m\cos\alpha\,\dot s\ ,$$

is preserved: $\dot P_X = 0$. The second equation of motion, corresponding to the generalized coordinate s, is

$$\bigg(1 + \frac{I}{ma^2}\bigg)\ddot s + \cos\alpha\,\ddot X = -g\,\sin\alpha\ .$$

Using conservation of P_X, we eliminate $\ddot s$ in favor of $\ddot X$, and immediately obtain

$$\ddot X = \frac{g\sin\alpha\cos\alpha}{\Big(1+\frac{M}{m}\Big)\Big(1+\frac{I}{ma^2}\Big) - \cos^2\alpha} \equiv a_X\ .$$

The result

$$\ddot s = -\frac{g\Big(1+\frac{M}{m}\Big)\sin\alpha}{\Big(1+\frac{M}{m}\Big)\Big(1+\frac{I}{ma^2}\Big) - \cos^2\alpha} \equiv a_s$$

follows immediately. Thus,

$$\begin{aligned} X(t) &= X(0) + \dot X(0)\,t + \tfrac{1}{2} a_X t^2 \\ s(t) &= s(0) + \dot s(0)\,t + \tfrac{1}{2} a_s t^2\ . \end{aligned}$$

Note that $a_s < 0$ while $a_X > 0$, *i.e.* the hoop rolls down and to the left as the wedge slides to the right. Note that $I = ma^2$ for a hoop; we've computed the answer here for general I.

8.6.4 Pendulum with nonrigid support

A particle of mass m is suspended from a flexible string of length ℓ in a uniform gravitational field. While hanging motionless in equilibrium, it is struck a horizontal blow resulting in an initial angular velocity ω_0. Treating the system as one with *two* degrees of freedom and a constraint, answer the following:

(a) Compute the Lagrangian, the equation of constraint, and the equations of motion.

Solution : The Lagrangian is

$$L = \tfrac{1}{2} m \left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + mgr\,\cos\theta \ .$$

The constraint is $r = \ell$. The equations of motion are

$$\begin{aligned} m\ddot{r} - mr\,\dot{\theta}^2 - mg\,\cos\theta &= \lambda \\ mr^2\,\ddot{\theta} + 2mr\,\dot{r}\,\dot{\theta} - mg\sin\theta &= 0 \ . \end{aligned}$$

(b) Compute the tension in the string as a function of angle θ.

Solution : Energy is conserved, hence

$$\tfrac{1}{2} m\ell^2\,\dot{\theta}^2 - mg\ell\cos\theta = \tfrac{1}{2} m\ell^2\,\dot{\theta}_0^2 - mg\ell\cos\theta_0 \ .$$

We take $\theta_0 = 0$ and $\dot{\theta}_0 = \omega_0$. Thus,

$$\dot{\theta}^2 = \omega_0^2 - 2\,\Omega^2\left(1 - \cos\theta\right) \ ,$$

with $\Omega = \sqrt{g/\ell}$. Substituting this into the equation for λ, we obtain

$$\lambda = mg\left\{2 - 3\cos\theta - \frac{\omega_0^2}{\Omega^2}\right\} \ .$$

(c) Show that if $\omega_0^2 < 2g/\ell$ then the particle's motion is confined below the horizontal and that the tension in the string is always positive (defined such that positive means exerting a pulling force and negative means exerting a pushing force). Note that the difference between a string and a rigid rod is that the string can only pull but the rod can pull or push. Thus, *the string tension must always be positive or else the string goes "slack"*.

Solution : Since $\dot{\theta}^2 \geq 0$, we must have

$$\frac{\omega_0^2}{2\Omega^2} \geq 1 - \cos\theta \ .$$

The condition for slackness is $\lambda = 0$, or

$$\frac{\omega_0^2}{2\Omega^2} = 1 - \tfrac{3}{2}\cos\theta \ .$$

Thus, if $\omega_0^2 < 2\Omega^2$, we have

$$1 > \frac{\omega_0^2}{2\Omega^2} > 1 - \cos\theta > 1 - \tfrac{3}{2}\cos\theta \ ,$$

and the string never goes slack. Note the last equality follows from $\cos\theta > 0$. The string rises to a maximum angle

$$\theta_{\text{max}} = \cos^{-1}\left(1 - \frac{\omega_0^2}{2\Omega^2}\right) \ .$$

(d) Show that if $2g/\ell < \omega_0^2 < 5g/\ell$ the particle rises above the horizontal and the string becomes slack (the tension vanishes) at an angle θ^*. Compute θ^*.

Solution : When $\omega^2 > 2\Omega^2$, the string rises above the horizontal and goes slack at an angle

$$\theta^* = \cos^{-1}\left(\tfrac{2}{3} - \frac{\omega_0^2}{3\Omega^2}\right) .$$

This solution craps out when the string is still taut at $\theta = \pi$, which means $\omega_0^2 = 5\Omega^2$.

(e) Show that if $\omega_0^2 > 5g/\ell$ the tension is always positive and the particle executes circular motion.

Solution : For $\omega_0^2 > 5\Omega^2$, the string never goes slack. Furthermore, $\dot{\theta}$ never vanishes. Therefore, the pendulum undergoes circular motion, albeit not with constant angular velocity.

8.6.5 Falling ladder

A uniform ladder of length ℓ and mass m has one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest and makes an angle θ_0 with respect to the horizontal.

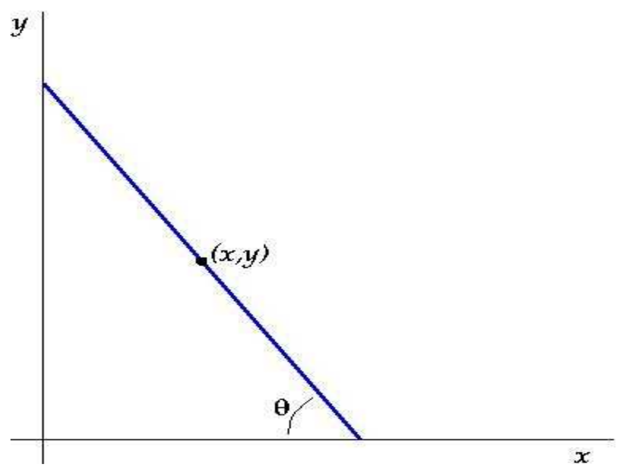

Figure 8.5: A ladder sliding down a wall and across a floor.

(a) Make a convenient choice of generalized coordinates and find the Lagrangian.

Solution : I choose as generalized coordinates the Cartesian coordinates (x, y) of the ladder's center of mass, and the angle θ it makes with respect to the floor. The Lagrangian is then

$$L = \tfrac{1}{2}\, m\,(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\, I\,\dot{\theta}^2 + mgy .$$

There are two constraints: one enforcing contact along the wall, and the other enforcing contact along the floor. These are written

$$
\begin{aligned}
G_1(x, y, \theta) &= x - \tfrac{1}{2}\,\ell\,\cos\theta = 0 \\
G_2(x, y, \theta) &= y - \tfrac{1}{2}\,\ell\,\sin\theta = 0\ .
\end{aligned}
$$

(b) Prove that the ladder leaves the wall when its upper end has fallen to a height $\frac{2}{3}L\sin\theta_0$.
The equations of motion are

$$
\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot q_\sigma}\bigg) - \frac{\partial L}{\partial q_\sigma} = \sum_j \lambda_j\,\frac{\partial G_j}{\partial q_\sigma}\ .
$$

Thus, we have

$$
\begin{aligned}
m\,\ddot x &= \lambda_1 = Q_x \\
m\,\ddot y + mg &= \lambda_2 = Q_y \\
I\,\ddot\theta &= \tfrac{1}{2}\ell\big(\lambda_1\sin\theta - \lambda_2\cos\theta\big) = Q_\theta\ .
\end{aligned}
$$

We now implement the constraints to eliminate x and y in terms of θ. We have

$$
\begin{aligned}
\dot x &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot\theta & \ddot x &= -\tfrac{1}{2}\,\ell\,\cos\theta\,\dot\theta^2 - \tfrac{1}{2}\,\ell\,\sin\theta\,\ddot\theta \\
\dot y &= \tfrac{1}{2}\,\ell\,\cos\theta\,\dot\theta & \ddot y &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot\theta^2 + \tfrac{1}{2}\,\ell\,\cos\theta\,\ddot\theta\ .
\end{aligned}
$$

We can now obtain the forces of constraint in terms of the function $\theta(t)$:

$$
\begin{aligned}
\lambda_1 &= -\tfrac{1}{2}m\ell\,\big(\sin\theta\,\ddot\theta + \cos\theta\,\dot\theta^2\big) \\
\lambda_2 &= +\tfrac{1}{2}m\ell\,\big(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\big) + mg\ .
\end{aligned}
$$

We substitute these into the last equation of motion to obtain the result

$$
I\,\ddot\theta = -I_0\,\ddot\theta - \tfrac{1}{2}mg\ell\,\cos\theta\ ,
$$

or

$$
(1+\alpha)\,\ddot\theta = -2\omega_0^2\,\cos\theta\ ,
$$

with $I_0 = \frac{1}{4}m\ell^2$, $\alpha \equiv I/I_0$ and $\omega_0 = \sqrt{g/\ell}$. This may be integrated once (multiply by $\dot\theta$ to convert to a total derivative) to yield

$$
\tfrac{1}{2}(1+\alpha)\,\dot\theta^2 + 2\,\omega_0^2\,\sin\theta = 2\,\omega_0^2\,\sin\theta_0\ ,
$$

which is of course a statement of energy conservation. This,

$$
\begin{aligned}
\dot\theta^2 &= \frac{4\,\omega_0^2\,(\sin\theta_0 - \sin\theta)}{1+\alpha} \\
\ddot\theta &= -\frac{2\,\omega_0^2\,\cos\theta}{1+\alpha}\ .
\end{aligned}
$$

We may now obtain $\lambda_1(\theta)$ and $\lambda_2(\theta)$:

$$\lambda_1(\theta) = -\frac{mg}{1+\alpha}\,\big(3\sin\theta - 2\sin\theta_0\big)\,\cos\theta$$

$$\lambda_2(\theta) = \frac{mg}{1+\alpha}\,\Big\{\big(3\sin\theta - 2\sin\theta_0\big)\,\sin\theta + \alpha\Big\}\ .$$

Demanding $\lambda_1(\theta) = 0$ gives the detachment angle $\theta = \theta_{\rm d}$, where

$$\sin\theta_{\rm d} = \tfrac{2}{3}\sin\theta_0\ .$$

Note that $\lambda_2(\theta_{\rm d}) = mg\alpha/(1+\alpha) > 0$, so the normal force from the floor is always positive for $\theta > \theta_{\rm d}$. The time to detachment is

$$T_1(\theta_0) = \int \frac{d\theta}{\dot\theta} = \frac{\sqrt{1+\alpha}}{2\,\omega_0}\int\limits_{\theta_{\rm d}}^{\theta_0}\frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}}\ .$$

(c) Show that the subsequent motion can be reduced to quadratures (*i.e.* explicit integrals).

Solution : After the detachment, there is no longer a constraint G_1. The equations of motion are

$$\begin{aligned} m\,\ddot x &= 0 \qquad \text{(conservation of x-momentum)}\\ m\,\ddot y + m\,g &= \lambda\\ I\,\ddot\theta &= -\tfrac{1}{2}\,\ell\,\lambda\,\cos\theta\ , \end{aligned}$$

along with the constraint $y = \frac{1}{2}\,\ell\,\sin\theta$. Eliminating y in favor of θ using the constraint, the second equation yields

$$\lambda = mg - \tfrac{1}{2}m\ell\,\sin\theta\,\dot\theta^2 + \tfrac{1}{2}m\ell\,\cos\theta\,\ddot\theta\ .$$

Plugging this into the third equation of motion, we find

$$I\,\ddot\theta = -2\,I_0\,\omega_0^2\,\cos\theta + I_0\,\sin\theta\,\cos\theta\,\dot\theta^2 - I_0\,\cos^2\theta\,\ddot\theta\ .$$

Multiplying by $\dot\theta$ one again obtains a total time derivative, which is equivalent to rediscovering energy conservation:

$$E = \tfrac{1}{2}\big(I + I_0\,\cos^2\theta\big)\,\dot\theta^2 + 2\,I_0\,\omega_0^2\,\sin\theta\ .$$

By continuity with the first phase of the motion, we obtain the initial conditions for this second phase:

$$\theta = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big)$$

$$\dot\theta = -2\,\omega_0\sqrt{\frac{\sin\theta_0}{3\,(1+\alpha)}}\ .$$

```
In[37]:= T[x_] := NIntegrate[√((4 / 3) / (x - Sin[y])) , {y, ArcSin[2 x / 3], ArcSin[x] - 10^-9}] / 2

In[38]:= S[x_] := NIntegrate[
            √((1 + (4 / 3) (Cos[y]) ^ 2) / ((1 - (x / 3) ^ 2) x - Sin[y])) , {y, 0, ArcSin[2 x / 3]}] / 2

In[39]:= Q[x_] := T[x] + S[x]

In[43]:= Plot[Q[x], {x, 0, 1}]
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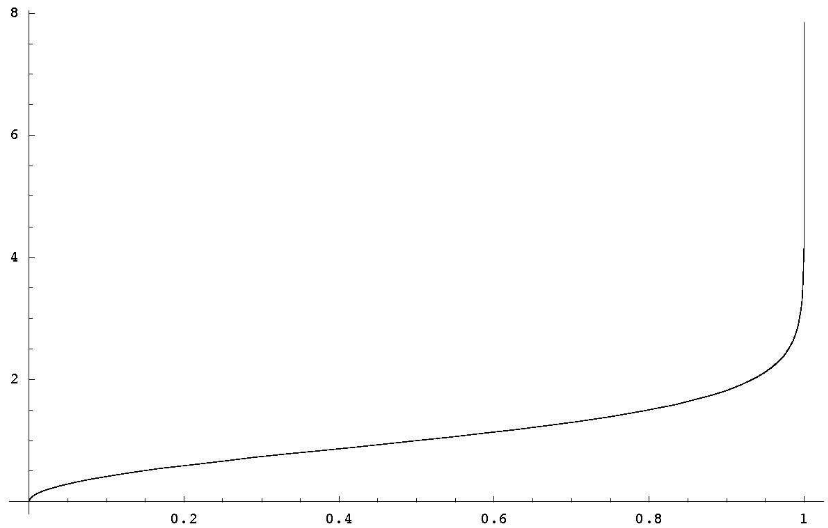

Figure 8.6: Plot of time to fall for the slipping ladder. Here $x = \sin\theta_0$.

Thus,

$$
\begin{aligned}
E &= \tfrac{1}{2}\big(I + I_0 - \tfrac{4}{9}\, I_0 \sin^2\theta_0\big) \cdot \frac{4\,\omega_0^2 \sin\theta_0}{3\,(1+\alpha)} + \tfrac{1}{3}\, mg\ell \sin\theta_0 \\
&= 2\, I_0\, \omega_0^2 \cdot \left\{ 1 + \tfrac{4}{27}\, \frac{\sin^2\theta_0}{1+\alpha} \right\} \sin\theta_0 \ .
\end{aligned}
$$

(d) Find an expression for the time $T(\theta_0)$ it takes the ladder to smack against the floor. Note that, expressed in units of the time scale $\sqrt{L/g}$, T is a dimensionless function of θ_0. Numerically integrate this expression and plot T *versus* θ_0.

Solution : The time from detachment to smack is

$$
T_2(\theta_0) = \int \frac{d\theta}{\dot\theta} = \frac{1}{2\,\omega_0} \int\limits_0^{\theta_{\rm d}} d\theta \, \sqrt{\frac{1 + \alpha \cos^2\theta}{\big(1 - \frac{4}{27}\frac{\sin^2\theta_0}{1+\alpha}\big)\sin\theta_0 - \sin\theta}} \ .
$$

The total time is then $T(\theta_0) = T_1(\theta_0) + T_2(\theta_0)$. For a uniformly dense ladder, $I = \frac{1}{12}\, m\ell^2 = \frac{1}{3}\, I_0$, so $\alpha = \frac{1}{3}$.

(e) What is the horizontal velocity of the ladder at long times?

Solution : From the moment of detachment, and thereafter,

$$\dot{x} = -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta} = \sqrt{\frac{4\,g\,\ell}{27\,(1+\alpha)}}\;\sin^{3/2}\!\theta_0 \ .$$

(f) Describe in words the motion of the ladder subsequent to it slapping against the floor.

Solution : Only a fraction of the ladder's initial potential energy is converted into kinetic energy of horizontal motion. The rest is converted into kinetic energy of vertical motion and of rotation. The slapping of the ladder against the floor is an elastic collision. After the collision, the ladder must rise again, and continue to rise and fall *ad infinitum*, as it slides along with constant horizontal velocity.

8.6.6 Point mass inside rolling hoop

Consider the point mass m inside the hoop of radius R, depicted in Fig. 8.7. We choose as generalized coordinates the Cartesian coordinates (X, Y) of the center of the hoop, the Cartesian coordinates (x, y) for the point mass, the angle ϕ through which the hoop turns, and the angle θ which the point mass makes with respect to the vertical. These six coordinates are not all independent. Indeed, there are only two independent coordinates for this system, which can be taken to be θ and ϕ. Thus, there are *four* constraints:

$$X - R\phi \equiv G_1 = 0 \tag{8.106}$$

$$Y - R \equiv G_2 = 0 \tag{8.107}$$

$$x - X - R\sin\theta \equiv G_3 = 0 \tag{8.108}$$

$$y - Y + R\cos\theta \equiv G_4 = 0 \ . \tag{8.109}$$

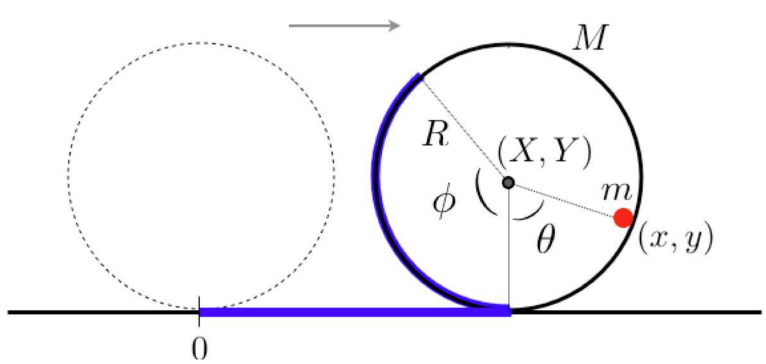

Figure 8.7: A point mass m inside a hoop of mass M, radius R, and moment of inertia I.

The kinetic and potential energies are easily expressed in terms of the Cartesian coordinates, aside from the energy of rotation of the hoop about its CM, which is expressed in terms of

$\dot{\phi}$:

$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 \tag{8.110}$$

$$U = MgY + mgy\ . \tag{8.111}$$

The moment of inertia of the hoop about its CM is $I = MR^2$, but we could imagine a situation in which I were different. For example, we could instead place the point mass inside a very short cylinder with two solid end caps, in which case $I = \frac{1}{2}MR^2$. The Lagrangian is then

$$L = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 - MgY - mgy\ . \tag{8.112}$$

Note that L as written is completely independent of θ and $\dot{\theta}$!

Continuous symmetry

Note that there is an continuous symmetry to L which is satisfied by all the constraints, under

$$\tilde{X}(\zeta) = X + \zeta \qquad \tilde{Y}(\zeta) = Y \tag{8.113}$$

$$\tilde{x}(\zeta) = x + \zeta \qquad \tilde{y}(\zeta) = y \tag{8.114}$$

$$\tilde{\phi}(\zeta) = \phi + \frac{\zeta}{R} \qquad \tilde{\theta}(\zeta) = \theta\ . \tag{8.115}$$

Thus, according to Noether's theorem, there is a conserved quantity

$$\begin{aligned} \Lambda &= \frac{\partial L}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{x}} + \frac{1}{R}\,\frac{\partial L}{\partial \dot{\phi}} \\ &= M\dot{X} + m\dot{x} + \frac{I}{R}\,\dot{\phi}\ . \end{aligned} \tag{8.116}$$

This means $\dot{\Lambda} = 0$. This reflects the overall conservation of momentum in the x-direction.

Energy conservation

Since neither L nor any of the constraints are explicitly time-dependent, the Hamiltonian is conserved. And since T is homogeneous of degree two in the generalized velocities, we have $H = E = T + U$:

$$E = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 + MgY + mgy\ . \tag{8.117}$$

Equations of motion

We have $n = 6$ generalized coordinates and $k = 4$ constraints. Thus, there are four undetermined multipliers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ used to impose the constraints. This makes for ten unknowns:

$$X \ , \ Y \ , \ x \ , \ y \ , \ \phi \ , \ \theta \ , \ \lambda_1 \ , \ \lambda_2 \ , \ \lambda_3 \ , \ \lambda_4 \ . \tag{8.118}$$

Accordingly, we have ten equations: six equations of motion plus the four equations of constraint. The equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \ . \tag{8.119}$$

Taking each generalized coordinate in turn, the equations of motion are thus

$$M\ddot{X} = \lambda_1 - \lambda_3 \tag{8.120}$$

$$M\ddot{Y} = -Mg + \lambda_2 - \lambda_4 \tag{8.121}$$

$$m\ddot{x} = \lambda_3 \tag{8.122}$$

$$m\ddot{y} = -mg + \lambda_4 \tag{8.123}$$

$$I\,\ddot{\phi} = -R\,\lambda_1 \tag{8.124}$$

$$0 = -R\cos\theta\,\lambda_3 - R\sin\theta\,\lambda_4 \ . \tag{8.125}$$

Along with the four constraint equations, these determine the motion of the system. Note that the last of the equations of motion, for the generalized coordinate $q_\sigma = \theta$, says that $Q_\theta = 0$, which means that the force of constraint on the point mass is radial. Were the point mass replaced by a rolling object, there would be an angular component to this constraint in order that there be no slippage.

Implementation of constraints

We now use the constraint equations to eliminate X, Y, x, and y in terms of θ and ϕ:

$$X = R\phi \quad , \quad Y = R \quad , \quad x = R\phi + R\sin\theta \quad , \quad y = R(1-\cos\theta) \ . \tag{8.126}$$

We also need the derivatives:

$$\dot{x} = R\,\dot{\phi} + R\cos\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\,\ddot{\phi} + R\cos\theta\,\ddot{\theta} - R\sin\theta\,\dot{\theta}^2 \ , \tag{8.127}$$

and

$$\dot{y} = R\sin\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\sin\theta\,\ddot{\theta} + R\cos\theta\,\dot{\theta}^2 \ , \tag{8.128}$$

as well as

$$\dot{X} = R\,\dot{\phi} \quad , \quad \ddot{X} = R\,\ddot{\phi} \quad , \quad \dot{Y} = 0 \quad , \quad \ddot{Y} = 0 \ . \tag{8.129}$$

We now may write the conserved charge as

$$\Lambda = \frac{1}{R}(I + MR^2 + mR^2)\,\dot{\phi} + mR\cos\theta\,\dot{\theta} \ . \tag{8.130}$$

This, in turn, allows us to eliminate $\dot{\phi}$ in terms of $\dot{\theta}$ and the constant Λ:

$$\dot{\phi} = \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR} - \dot{\theta}\cos\theta\right) \ , \tag{8.131}$$

where

$$\gamma = \frac{mR^2}{I + MR^2} \ . \tag{8.132}$$

The energy is then

$$\begin{aligned} E &= \tfrac{1}{2}(I + MR^2)\,\dot{\phi}^2 + \tfrac{1}{2}m\big(R^2\,\dot{\phi}^2 + R^2\,\dot{\theta}^2 + 2R^2\cos\theta\,\dot{\phi}\,\dot{\theta}\big) + MgR + mgR(1-\cos\theta) \\ &= \tfrac{1}{2}mR^2\left\{\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) + \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR}\right)^2 + \frac{2Mg}{mR}\right\} \ . \end{aligned} \tag{8.133}$$

The last two terms inside the big bracket are constant, so we can write this as

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) = \frac{4gk}{R} \ . \tag{8.134}$$

Here, k is a dimensionless measure of the energy of the system, after subtracting the aforementioned constants. If $k > 1$, then $\dot{\theta}^2 > 0$ for all θ, which would result in 'loop-the-loop' motion of the point mass inside the hoop – provided, that is, the normal force of the hoop doesn't vanish and the point mass doesn't detach from the hoop's surface.

Equation motion for $\theta(t)$

The equation of motion for θ obtained by eliminating all other variables from the original set of ten equations is the same as $\dot{E} = 0$, and may be written

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\ddot{\theta} + \left(\frac{\gamma\sin\theta\cos\theta}{1+\gamma}\right)\dot{\theta}^2 = -\frac{g}{R} \ . \tag{8.135}$$

We can use this to write $\ddot{\theta}$ in terms of $\dot{\theta}^2$, and, after invoking eqn. 8.134, in terms of θ itself. We find

$$\dot{\theta}^2 = \frac{4g}{R}\cdot\left(\frac{1+\gamma}{1+\gamma\sin^2\!\theta}\right)\big(k - \sin^2\!\tfrac{1}{2}\theta\big) \tag{8.136}$$

$$\ddot{\theta} = -\frac{g}{R}\cdot\frac{(1+\gamma)\sin\theta}{\big(1+\gamma\sin^2\!\theta\big)^2}\,\Big[4\gamma\,\big(k - \sin^2\!\tfrac{1}{2}\theta\big)\cos\theta + 1 + \gamma\sin^2\!\theta\Big] \ . \tag{8.137}$$

Forces of constraint

We can solve for the λ_j, and thus obtain the forces of constraint $Q_\sigma = \sum_j \lambda_j \frac{\partial G_j}{\partial q_\sigma}$.

$$\begin{aligned}\lambda_3 &= m\ddot{x} = mR\,\ddot{\phi} + mR\cos\theta\,\ddot{\theta} - mR\sin\theta\,\dot{\theta}^2 \\ &= \frac{mR}{1+\gamma}\Big[\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta\Big]\end{aligned} \tag{8.138}$$

$$\begin{aligned}\lambda_4 &= m\ddot{y} + mg = mg + mR\sin\theta\,\ddot{\theta} + mR\cos\theta\,\dot{\theta}^2 \\ &= mR\Big[\ddot{\theta}\,\sin\theta + \dot{\theta}^2\sin\theta + \frac{g}{R}\Big]\end{aligned} \tag{8.139}$$

$$\lambda_1 = -\frac{I}{R}\,\ddot{\phi} = \frac{(1+\gamma)I}{mR^2}\,\lambda_3 \tag{8.140}$$

$$\lambda_2 = (M+m)g + m\ddot{y} = \lambda_4 + Mg\ . \tag{8.141}$$

One can check that $\lambda_3\cos\theta + \lambda_4\sin\theta = 0$.

The condition that the normal force of the hoop on the point mass vanish is $\lambda_3 = 0$, which entails $\lambda_4 = 0$. This gives

$$-(1+\gamma\sin^2\!\theta)\,\cos\theta = 4(1+\gamma)\left(k - \sin^2\!\tfrac{1}{2}\theta\right)\ . \tag{8.142}$$

Note that this requires $\cos\theta < 0$, *i.e.* the point of detachment lies above the horizontal diameter of the hoop. Clearly if k is sufficiently large, the equality cannot be satisfied, and the point mass executes a periodic 'loop-the-loop' motion. In particular, setting $\theta = \pi$, we find that

$$k_{\rm c} = 1 + \frac{1}{4(1+\gamma)}\ . \tag{8.143}$$

If $k > k_{\rm c}$, then there is periodic 'loop-the-loop' motion. If $k < k_{\rm c}$, then the point mass may detach at a critical angle θ^*, but only if the motion allows for $\cos\theta < 0$. From the energy conservation equation, we have that the maximum value of θ achieved occurs when $\dot{\theta} = 0$, which means

$$\cos\theta_{\rm max} = 1 - 2k\ . \tag{8.144}$$

If $\frac{1}{2} < k < k_{\rm c}$, then, we have the possibility of detachment. This means the energy must be large enough but not too large.