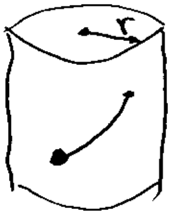
PHYSICS 110A SOLUTION SET 4

1)



$$ds^2 = r^2 d\theta^2 + dz^2$$

$$L[\theta(z)] = \int ds = \int dz \left(\sqrt{r^2 \left(\frac{d\theta}{dz}\right)^2 + 1} \right.$$

$$\Rightarrow \frac{d}{dz}\frac{\partial}{\partial\theta'}\left[\sqrt{r^2\theta'^2+1}\right] = \frac{\partial}{\partial\theta}\left[\sqrt{r^2\theta'^2+1}\right]$$

$$\Rightarrow \frac{\partial}{\partial\theta'}\left[\sqrt{r^2\theta'^2+1}\right] = \text{CONST} = C_1$$

$$\therefore \quad \frac{r^2\theta'}{\sqrt{r^2\theta'^2+1}} = C_1$$

$$r^4\theta'^2 = C_1^2\left(r^2\theta'^2+1\right)$$

$$\theta' = \sqrt{\frac{C_1^2}{r^4 - C_1^2 r^2}}$$

$$\therefore \underline{\theta = \theta_0 + \zeta z} \quad , \text{with } \sqrt{\frac{C_1^2}{r^4 - C_1^2 r^2}} \equiv \zeta$$

2) WE WANT TO MAXIMIZE $V = \pi R^2 L$ SUBJECT TO THE CONSTRAINT $G(R,L) = 2\pi R + \frac{L^2}{a} - b = 0$

$$\mathcal{L}^* = V + \lambda G$$

THUS: $\frac{\partial \mathcal{L}^*}{\partial R} = 0 \Rightarrow 2\pi R L + \lambda 2\pi = 0$

$$\frac{\partial \mathcal{L}^*}{\partial L} = 0 \Rightarrow \pi R^2 + \frac{2L}{a}\lambda = 0$$

$$\frac{\partial \mathcal{L}^*}{\partial \lambda} = 0 \Rightarrow 2\pi R + \frac{L^2}{a} - b = 0$$

FROM (R) : $\lambda = -RL$

THUS: $\pi R^2 - \frac{2RL^2}{a} = 0$

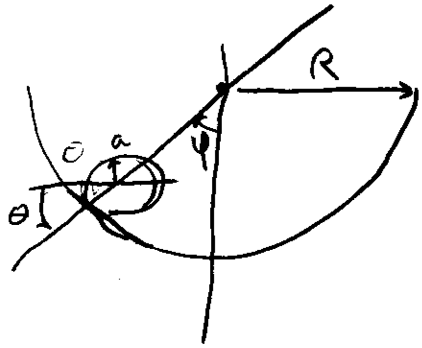
$$R = \frac{2}{a\pi} L^2$$

THUS, (G) GIVES US: $\frac{4}{a} L^2 + \frac{L^2}{a} = b$

$L = \sqrt{\frac{ba}{5}}$, AND $R = \frac{2}{a\pi}\left(\frac{ba}{5}\right) = \frac{2b}{5\pi}$

$$\therefore \boxed{V = \pi R^2 L = \frac{4b^2}{25\pi}\sqrt{\frac{ba}{5}}}$$

3)



CONSTRAINTS: $R\varphi = a\theta$

$$U = mgy = mg\left((R-a) - (R-a)\cos\varphi\right)$$

$$T = \frac{1}{2}m(R-a)^2\dot{\varphi}^2 + \frac{1}{2}I\dot{\theta}^2$$

$\therefore$

$$L = T - U = \frac{1}{2}m(R-a)^2\dot{\varphi}^2 + \frac{1}{2}I\left(\frac{a}{R}\right)^2\dot{\varphi}^2 + mg(R-a)\cos\varphi$$

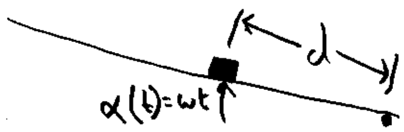
$$(\varphi):\ m(R-a)^2\ddot{\varphi} + I\left(\frac{a}{R}\right)^2\ddot{\varphi} + mg(R-a)\sin\varphi = 0$$

SMALL $\measuredangle$ APPROX:

$$\ddot{\varphi} + \frac{mg(R-a)}{m(R-a)^2 + I\left(\frac{a}{R}\right)^2}\varphi = 0$$

$$\therefore\ \omega^2 = \frac{mg(R-a)}{m(R-a)^2 + I\left(\frac{a}{R}\right)^2}$$

4)



$$x = d\cos\omega t$$
$$y = d\sin\omega t$$

$$L = T - U = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgd\sin\omega t$$
$$= \frac{m}{2}(\dot{d}^2 + d^2\omega^2) - mgd\sin\omega t$$

$$(d):\ m\ddot{d} - md\omega^2 + mg\sin\omega t = 0$$

$$m\ddot{d} - md\omega^2 = mg\sin\omega t$$
$$\ddot{d} - d\omega^2 = g\sin\omega t$$

WE ARE NOT GROWING

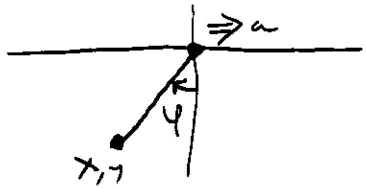
GENERAL SOLUTION: $d = \cancel{Ae^{\omega t}} + Be^{-\omega t}$

PARTICULAR SOLUTION: $-\omega^2 A - \omega^2 = g$

$$A = -\frac{g}{\omega^2} - 1$$

$\therefore$ SOLUTION IS: $d = d_0 e^{-\omega t} - \left(\frac{g}{\omega^2} - 1\right)\sin\omega t, \quad \omega t < \frac{\pi}{2}$

5)



$x = \frac{1}{2}at^2 - l\sin\varphi \qquad \dot{x} = at - l\dot{\varphi}\cos\varphi$

$y = l(1-\cos\varphi) \qquad \dot{y} = l\dot{\varphi}\sin\varphi$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\left(a^2t^2 - 2atl\dot{\varphi}\cos\varphi + l^2\dot{\varphi}^2\right)$

$U = mgy = mgl(1-\cos\varphi)$

$\therefore \mathcal{L} = \frac{1}{2}m\left(a^2t^2 - 2atl\dot{\varphi}\cos\varphi + l^2\dot{\varphi}^2\right) - mgl(1-\cos\varphi)$

$(\varphi): \frac{d}{dt}\left(-matl\cos\varphi + ml^2\dot{\varphi}\right) + mgl\sin\varphi - matl\dot{\varphi}\cos\varphi$

$= mal\cos\varphi + \cancel{matl\dot{\varphi}\sin\varphi} + ml^2\ddot{\varphi} + mgl\sin\varphi - \cancel{matl\dot{\varphi}\cos\varphi} = 0$

$ml^2\ddot{\varphi} + mgl\sin\varphi + mal\cos\varphi = 0$

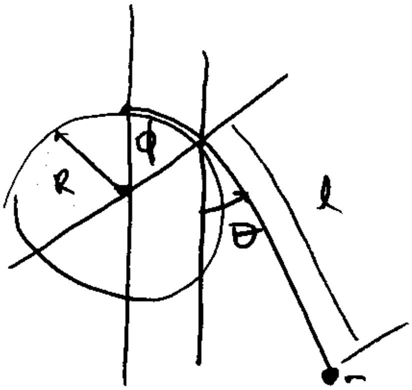
$ml^2\ddot{\varphi} + ml(g\sin\varphi + a\cos\varphi) = 0$

$\Rightarrow \underline{ml^2\ddot{\varphi} + ml\sqrt{a^2+g^2}\,\sin\left(\varphi + \tan^{-1}\frac{a}{g}\right) = 0}$ THIS LOOKS LIKE A CONFORMITY TO ME. ; $\omega^2 = \frac{\sqrt{a^2+g^2}}{l}$

FOR VERTICAL ACCELLERATION, WE GET

$ml^2\ddot{\varphi} + ml(g+a)\sin\varphi = 0$

(6)



CONSTRAINTS: $\theta = \frac{\pi}{2} - \phi$

$l + R\phi = L = \text{CONST}$

$$y = \pm R(1-\cos\phi) + l\cos\theta \; ; \; \dot{y} = R\dot{\phi}\sin\phi + \dot{l}\cos\theta - l\dot{\theta}\cos\theta$$

$$x = R\sin\phi + l\sin\theta \; ; \; \dot{x} = R\dot{\phi}\cos\phi + \dot{l}\sin\theta + l\dot{\theta}\cos\theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(R^2\dot{\phi}^2 + R\dot{\phi}\dot{l}\sin(\phi+\theta) - R\dot{\phi}l\dot{\theta}\cos(\theta-\phi) + \dot{l}^2 + l^2\dot{\theta}^2\right)$$

$$U = -mgy$$

$$\therefore \mathcal{L}^* = T - U + \sum \lambda_i G_i$$
$$= \frac{1}{2}m\left[R^2\dot{\phi}^2 + R\dot{\phi}\dot{l}\sin(\phi+\theta) - R\dot{\phi}l\dot{\theta}\cos(\theta-\phi) + \dot{l}^2 + l^2\dot{\theta}^2\right]$$
$$+ mg\left[R(1-\cos\phi) + l\cos\theta\right] + \lambda_1\left(\theta + \phi - \frac{\pi}{2}\right) + \lambda_2(l + R\phi - L)$$

EQUATIONS OF MOTION:

LET'S APPLY G_1 BEFORE ANYTHING ELSE:

$\dot{\theta} = -\dot{\phi}$; $\theta = \frac{\pi}{2} - \phi$

$$\mathcal{L}^* = \frac{1}{2}m\left[R^2\dot{\phi}^2 + R\dot{\phi}\dot{l} + Rl\dot{\phi}^2\sin 2\phi - \quad + \dot{l}^2 + l^2\dot{\phi}^2\right]$$
$$+ mg\left[R(1-\cos\phi) + l\sin\phi\right] + \lambda(l + R\phi - L)$$

λ: $l + R\phi = L$

ϕ: $mR^2\ddot{\phi} + \frac{m}{2}R\ddot{l} + R\dot{l}\dot{\phi}\sin 2\phi + 2Rl\dot{\phi}^2\cos 2\phi + l^2\ddot{\phi} + 2l\dot{l}\dot{\phi}$
$- mgR\sin\phi - mgl\cos\phi - \lambda(R) = 0$

l: $\frac{1}{2}mR\ddot{\phi} + m\ddot{l} - mg\sin\phi + \lambda = 0$

LINEARIZE ABOUT $\phi \approx \frac{\pi}{2}$, $\dot{l} \approx l_0$, AND NEGLECT CONSTANT TERMS.

$$\lambda: \ \delta l + R\delta\phi \approx 0$$

$$\phi: \ mR^2\delta\ddot{\phi} + \frac{mR}{2}\delta\ddot{l} - \cancel{mgR} = 0$$

$$l: \ \tfrac{1}{2}mR\,\delta\ddot{\phi} + m\delta\ddot{l} = 0$$

SORRY - I GOOFED SOMEWHERE. IF YOU WANT TO SEE A COMPLETE SOLUTION, SEE ME LATER.

---

(7) $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$$2x\dot{x} + 2y\dot{y} = 0$$

$$\dot{y} = \frac{-x}{y}\dot{x}$$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \frac{x^2}{x^2 - l^2}\dot{x}^2\right) - mg\left(\sqrt{l^2 - x^2}\right)$$

$$(x): \ m\ddot{x} + \cancel{\frac{mx\dot{x}^3}{x^2-l^2}} - \cancel{\frac{mx^3\dot{x}}{(x^2-l^2)^2}} - \dot{x}^2\left[\cancel{\frac{mx}{x^2-l^2}} + \cancel{\frac{mx^3}{(x^2-l^2)^2}}\right] + \frac{mgx}{\sqrt{l^2-x^2}}$$

$$\delta\ddot{x} + \frac{g}{l}\delta x = 0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(x^2 + y^2 - l^2)$$

$$\lambda: \ x^2 + y^2 = l^2$$

$$x: \ m\ddot{x} - 2\lambda x = 0$$

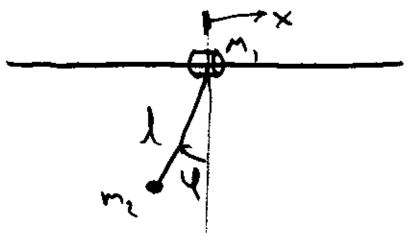
$$y: \ m\ddot{y} + mg - 2\lambda y = 0 \Rightarrow \lambda = \frac{+m\ddot{y} + mg}{2y}$$

$$\therefore \ \ddot{x} + x\left[\frac{-\ddot{y} + g}{y}\right] = 0 \qquad y = \sqrt{l^2 - x^2}$$

BUT $\ddot{y} \approx 0$ /* LOTS OF CROSS TERMS */

SO, AS ABOVE, $\delta\ddot{x} + \frac{g}{l}x = 0$

(8)



$$x_1 = x + l\sin\varphi$$
$$y_1 = l\cos\varphi$$

$$\mathcal{L} = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \left(\dot{x}^2 + 2\dot{x}\dot{\varphi} l\cos\varphi + \dot{l}^2\dot{\varphi}^2\right) + mgl\cos\varphi$$

SINCE $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial \dot{x}} = \text{CONST} \Rightarrow m_1\dot{x} + m_2\dot{x} + 2\dot{\varphi} l\cos\varphi\, m_2 = C$

$$(\varphi):\ m_2 l^2\ddot{\varphi} + m_2\ddot{x} l\cos\varphi - \cancel{m_2\dot{x}\dot{\varphi}} l\sin\varphi + \cancel{m_2\dot{x}\dot{\varphi}} l\sin\varphi + m_2 g l\sin\varphi = 0$$

$$m_2 l^2\ddot{\varphi} + m_2 l\cos\varphi \frac{m_2}{(m_1+m_2)} 2\left(\ddot{\varphi} l\cos\varphi - \dot{\varphi}^2 l\sin\varphi\right) + m_2 g l\sin\varphi = 0$$

$$\left[m_2 l^2 - \frac{2m_2^2 l^2}{m_1+m_2}\right]\delta\ddot{\varphi} + m_2 g l\,\delta\varphi = 0$$

SO, FOR SMALL OSCILLATIONS, $\omega^2 = \dfrac{g}{l\left(1 - \frac{2m_2}{m_1+m_2}\right)}$

THIS DESCRIBES A SYSTEM IN WHICH THE CENTER OF MASS DOES NOT MOVE (IN THE X DIRECTION) AND BOTH MASSES OSCILLATE BACK AND FORTH ACCORDINGLY.