PHYSICS 110A : CLASSICAL MECHANICS
FALL 2007 FINAL EXAM

[1] Two masses and two springs are configured linearly and externally driven to rotate with angular velocity ω about a fixed point, as shown in fig. 1. The unstretched length of each spring is a.

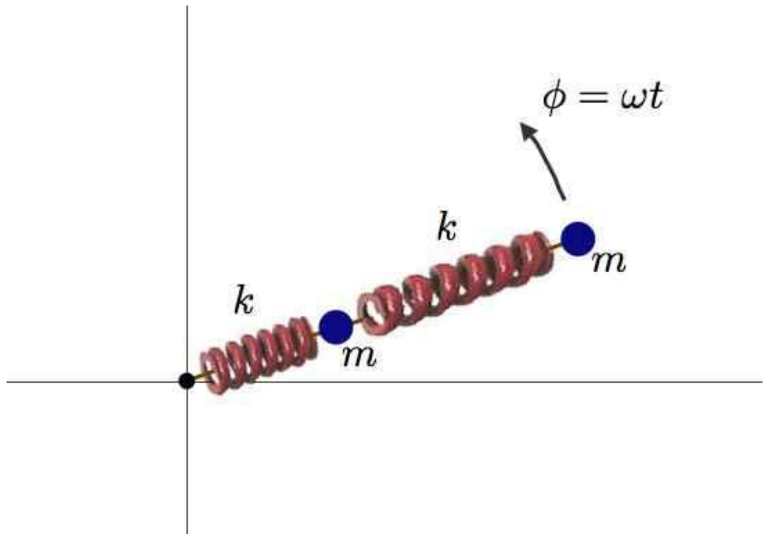

Figure 1: Two masses and two springs rotate with angular velocity ω.

`(a)` Choose as generalized coordinates the radial distances $r_{1,2}$ from the origin. Find the Lagrangian $L(r_1, r_2, \dot{r}_1, \dot{r}_2, t)$.
`[5 points]`

`(b)` Derive expressions for all conserved quantities.
`[5 points]`

`(c)` What equations determine the equilibrium radii r_1^0 and r_2^0? (You do not have to solve these equations.)
`[5 points]`

`(d)` Suppose now that the system is not externally driven, and that the angular coordinate ϕ is a dynamical variable like r_1 and r_2. Find the Lagrangian $L(r_1, r_2, \phi, \dot{r}_1, \dot{r}_2, \dot{\phi}, t)$.
`[5 points]`

`(e)` For the system described in part `(d)`, find expressions for all conserved quantities.
`[5 points]`

[**2**] A point mass m slides inside a hoop of radius R and mass M, which itself rolls without slipping on a horizontal surface, as depicted in fig. 2.

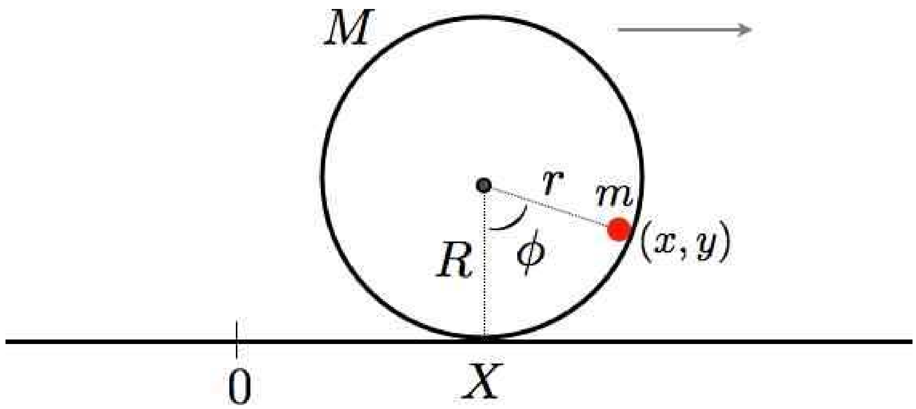

Figure 2: A mass point m rolls inside a hoop of mass M and radius R which rolls without slipping on a horizontal surface.

Choose as general coordinates (X, ϕ, r), where X is the horizontal location of the center of the hoop, ϕ is the angle the mass m makes with respect to the vertical ($\phi = 0$ at the bottom of the hoop), and r is the distance of the mass m from the center of the hoop. Since the mass m slides inside the hoop, there is a constraint:

$$G(X, \phi, r) = r - R = 0 \ .$$

Nota bene: The kinetic energy of the moving hoop including translational and rotational components (but not including the mass m),, is $T_{\text{hoop}} = M\dot{X}^2$ (*i.e.* twice the translational contribution alone).

`(a)` Find the Lagrangian $L(X, \phi, r, \dot{X}, \dot{\phi}, \dot{r}, t)$.
`[5 points]`

`(b)` Find *all* the generalized momenta p_σ, the generalized forces F_σ, and the forces of constraint Q_σ.
`[10 points]`

`(c)` Derive expressions for all conserved quantities.
`[5 points]`

`(d)` Derive a differential equation of motion involving the coordinate $\phi(t)$ alone. *I.e.* your equation should not involve r, X, or the Lagrange multiplier λ.
`[5 points]`

[3] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|^{1/4}$.

(a) Sketch the effective potential $U_{\text{eff}}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.
[5 points]

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?
[5 points]

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?
[5 points]

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.
[5 points]

(e) What value of n would result in a perturbed orbit shaped like that in fig. 3?
[5 points]

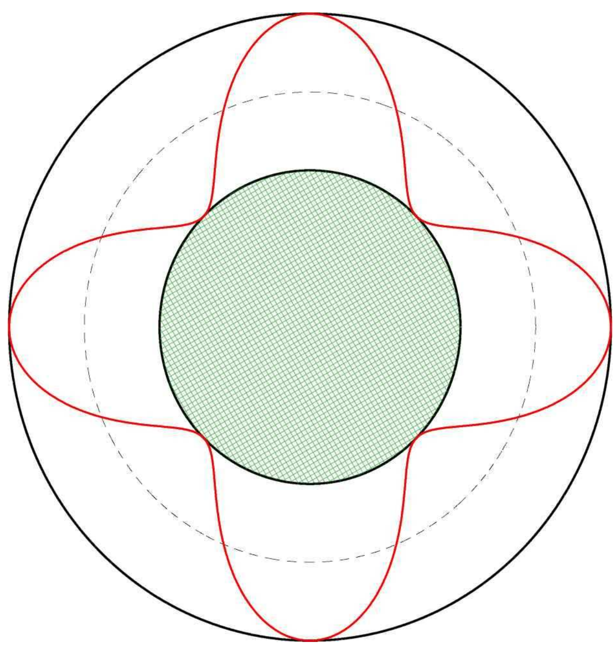

Figure 3: Closed precession in a central potential $U(r) = kr^n$.

[**4**] Two masses and three springs are arranged as shown in fig. 4. You may assume that in equilibrium the springs are all unstretched with length a. The masses and spring constants are simple multiples of fundamental values, *viz.*

$$m_1 = m \quad , \quad m_2 = 4\,m \quad , \quad k_1 = k \quad , \quad k_2 = 4k \quad , \quad k_3 = 28k \ . \tag{1}$$

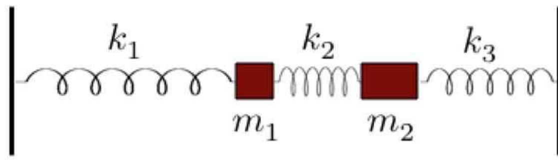

Figure 4: Coupled masses and springs.

`(a)` Find the Lagrangian.
`[5 points]`

`(b)` Find the T and V matrices.
`[5 points]`

`(c)` Find the eigenfrequencies ω_1 and ω_2.
`[5 points]`

`(d)` Find the modal matrix $\mathrm{A}_{\sigma i}$.
`[5 points]`

`(e)` Write down the most general solution for the motion of the system.
`[5 points]`