PHYSICS 110A : CLASSICAL MECHANICS
PROBLEM SET #3

[1] Prove that the shortest distance between two points in Euclidean space of arbitrary dimension is a straight line.

[2] In the lecture notes I discussed the example of a geodesic on a surface of revolution $z = z(\rho)$. The distance functional was written there as

$$\begin{aligned} D\big[\phi(\rho)\big] &= \int \sqrt{d\rho^2 + \rho^2\, d\phi^2 + dz^2} \\ &= \int\limits_{\rho_1}^{\rho_2} d\rho\, \sqrt{1 + z'^2(\rho) + \rho^2\, \phi'^2(\rho)}\ , \end{aligned} \tag{1}$$

from which the Euler-Lagrange equation for $\phi(\rho)$ follows. I further showed that the integrand was a function $L(\phi, \phi', \rho)$ which was independent of ϕ, in which case the Euler-Lagrange equations give us that $\partial L/\partial \phi'$ is a constant. In this treatment I have regarded ρ as the independent variable, and the geodesic is a function $\phi(\rho)$.

(a) Suppose instead we regard ϕ as the independent variable, in which case the geodesic is a function $\rho(\phi)$. Show that the distance functional may be written as

$$D\big[\rho(\phi)\big] = \int\limits_{\phi_1}^{\phi_2} d\phi\, \sqrt{\rho^2 + \big[1 + z'^2(\rho)\big]\bigg(\frac{d\rho}{d\phi}\bigg)^2}\ , \tag{2}$$

and find the Euler-Lagrange equations.

(b) The integrand is now of the form $L(\rho, \rho', \phi)$ with $\partial L/\partial \phi = 0$, *i.e.* there is no appearance of the *independent variable* ϕ in L. Consult eqns. 6.25-26 of the notes to see what happens when L is cyclic in the independent variable, and find the resulting first integral $\mathcal{J}$.

(c) Show that the first order equation which results for $\rho(\phi)$ is equivalent to the first order equation 6.43 for $\phi(\rho)$.

[3] Describe the plane paths of light in (two-dimensional) media in which the index of refraction varies as: *(a)* $n = ay$, *(b)* $n = ay^{-1}$, *(c)* $n = ay^{1/2}$, and *(d)* $n = ay^{-1/2}$, where $y > 0$ and where in each case a is a positive constant.

[4] Consider the system of masses and springs shown in Fig. 1. All motion is horizontal and frictionless.

(a) Choose a set of generalized coordinates, and find T, U, and L.

(b) Find the equations of motion.

(c) For the brave only! Find the natural frequencies of this system.

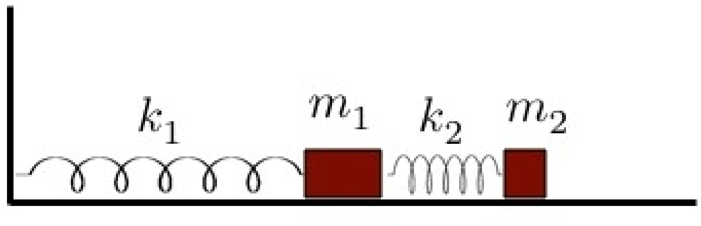

Figure 1: Two masses connected by two springs.

[5] A particle of mass m moves in a two-dimensional plane under the influence of a potential

$$U(x,y) = U_0 \ln\left(\frac{x^2+y^2}{a^2}\right) , \tag{3}$$

where a is a constant.

(a) Make an intelligent choice for a set of generalized coordinates and find T, U, and L.

(b) Write down the equations of motion.

(c) What, if any, conserved quantities exist?