→ How → Dynamics of Spiral Wave Growth and Maintenance ?

Essence of problem is how spiral waves are → maintained (i.e. extract free energy from mean profiles in disk)
→ excited ⟺ resonance interaction
→ executing angular momentum transport

∴ need develop wave kinetics for disks.
⇒ a) wave energy and momentum
b.) negative energy spiral modes ⟺ physics
c.) eikonal theory and wave kinetics

Few wave energy, angular momentum:

$$\frac{dE_w}{dt} = \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr\, \tilde{\sigma}\, \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

consider adiabatic turn-on of external perturbation, from $t = -\infty$.

i.e. apply $\tilde{\phi}_{ext}$

$$\frac{dE}{dt} = \frac{\partial \phi_{ext}}{\partial t}$$

so

$$E_{wave} = \int_{-\infty}^{t} dt' \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr\, \tilde{\sigma} \left(\frac{\partial \tilde{\phi}}{\partial t}\right)$$

Likewise

$$L_{\Theta\, \text{wave}} = \int_{-\infty}^{t} dt' \int_0^{\infty} r\,dr \int_0^{2\pi} d\phi\, \tilde{\sigma}\, r \left(- \frac{\partial \tilde{\phi}_{ext}}{r\,\partial\phi} \right)$$

↳ torque density (external)

$$= \int_{-\infty}^{t} dt' \int_0^{\infty} r\,dr \int_0^{2\pi} d\phi \left(-\tilde{\sigma} \frac{\partial \tilde{\phi}_{ext}}{\partial \phi} \right)$$

Now, need relate external perturbation to self-consistent field, i.e. aka dielectric for plasma

$$\nabla^2 \underset{\text{plasma field}}{\phi} = -4\pi \left(\tilde{\rho}_{induced} + \underset{\text{external probe}}{\tilde{\rho}_{ext}} \right)$$

$$\underset{\text{dielectric}}{\epsilon(k,\omega)}\, \phi_{k,\omega} = \tilde{\rho}^{\,ext}_{k,\omega} \frac{4\pi}{k^2}$$

$$\underset{\text{dielectric}}{\epsilon} = 1 + 4\pi \underset{\text{susceptibility}}{\chi(k,\omega)}$$

i.e. dynamic collective response

so

$$\nabla^2 \phi = 4\pi G \sigma\, \delta(z)$$

$$+ \nabla^2 \tilde{\phi} - 4\pi G \sigma_0 \frac{\tilde{\sigma}^{ind}}{\sigma_0} \delta(z) = 4\pi G \sigma_0 \frac{\tilde{\sigma}_{ext}}{\sigma_0} \delta(z)$$

$$\int_{-\epsilon}^{+\epsilon} dz \Rightarrow$$

$$-2|k|\tilde{\phi}_k - 4\pi G\Sigma_0 \frac{\tilde{\Sigma}_k}{\Sigma_0} = 4\pi G\Sigma_0 \frac{\tilde{\Sigma}_{ext}}{\Sigma_0}$$

$$\frac{\tilde{\Sigma}_{k,\omega}}{\Sigma_0} = \underbrace{\hat{\chi}_{grav}(k,\omega)}_{\text{gravitational susceptibility}} \tilde{\phi}_{k,\omega}$$

$$\left[|k| + 2\pi G\Sigma_0 \hat{\chi}(k,\omega)\right] \tilde{\phi}_k = -2\pi G\Sigma_0 \frac{\tilde{\Sigma}_{ext}}{\Sigma_0}$$

$$\Rightarrow \left[1 + \frac{2\pi G\Sigma_0}{|k|} \hat{\chi}(k,\omega)\right] \tilde{\phi}_k = -\frac{2\pi G\Sigma_0}{|k|} \frac{\tilde{\Sigma}_{ext}}{\Sigma_0}$$

$$\therefore \underbrace{D(k,\omega)}_{\epsilon^{eff}_{k,\omega}} = 1 + \frac{2\pi G\Sigma_0}{|k|} \hat{\chi}(k,\omega)$$ $\rightsquigarrow$ effective dielectric

$$\hat{\chi} = \delta(\tilde{\Sigma}/\Sigma_0)/\delta\phi$$ $\rightsquigarrow$ susceptibility

For $\mathcal{D}(k,\omega)$ ⇒ relate Σ to $\tilde{\phi}$

⇒

$$\left.\begin{aligned} -i\bar{\omega}\,\tilde{V}_r - 2\Omega\,\tilde{V}_\theta &= -ik_r\left(c_s^2\,\tilde{\Sigma}/\Sigma_0 + \tilde{\phi}\right) \\ -i\omega\,\tilde{V}_\theta + \frac{\tilde{V}_r}{r}\frac{\partial}{\partial r}(r^2\Omega) &= 0 \end{aligned}\right\}\ \text{tight winding}$$

$$-i\bar{\omega}\tilde{\Sigma}_{k,\omega} + ik_r\Sigma_0\tilde{V}_r = 0$$

⇒

$$\frac{\tilde{\Sigma}}{\Sigma_0}\left(1 - \frac{k_r^2 c_s^2}{\bar{\omega}^2 - \kappa^2}\right) = \frac{k_r^2\tilde{\phi}}{\bar{\omega}^2 - \kappa^2}$$

and

$$\tilde{\Sigma}/\Sigma_0 = k_r^2\tilde{\phi}\Big/\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2$$

∴

$$\boxed{\begin{aligned} \chi(k,\omega) &= k_r^2\Big/\left(\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2\right) \\ \mathcal{D}(k,\omega) &= 1 + \frac{2\pi G\Sigma_0 |k_r|}{\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2} \end{aligned}}$$

$$\mathcal{D}(k,\omega) = 0 \quad \Rightarrow \quad \bar{\omega}^2 = \kappa^2 + k_r^2 c_s^2 - 2\pi G\Sigma_0|k_r|$$

recovers Lin-Shu.

Exercise: Re-visit in kinetics

Now, if include dissipation (why? → need infinitesimal dissipn. to get irreversibility for $\int_{-\infty}^{t}$), recall:

$$\epsilon(k,\omega) = \epsilon_r(k,\omega) + i\,\epsilon_{IM}(k,\omega)$$

$$\epsilon(k,\omega) = 0 \Rightarrow$$

$$\epsilon_r(k,\omega_{k}) + \underbrace{(\omega-\omega_k)}_{i\gamma_k}\left.\frac{\partial \epsilon_r}{\partial \omega}\right|_{\omega_k} = -i\,\epsilon_{IM}(k,\omega_k)$$

$$\gamma_k = -\epsilon_{IM}(k,\omega_k)\Big/\left.\partial\epsilon/\partial\omega\right|_{\omega_k}$$

$\partial\epsilon/\partial\omega > 0 \Rightarrow \epsilon_{IM} > 0 \to$ damping, $< 0 \to$ growth (usual waves)

$< 0 \Rightarrow \epsilon_{IM} > 0 \to$ growth, $< 0 \to$ damping (negative energy)

∴ suggests that sign $\partial D/\partial\omega$ is crucial for spiral wave dynamics (i.e. ⊕, ⊖ negative energy wave)

Now,
$$\frac{dE_W}{dt} = \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr\, \tilde{\sigma}\, \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

but $\quad \Delta(\mathbf{k},\omega)\,\hat{\phi}_{k,\omega} = \hat{\phi}^{ext}_{k,\omega}$

near/at collective ~~resonance~~: (mode j)
resonance

$$\Delta(\mathbf{k},\omega) = \cancel{\Delta(\mathbf{k},\omega_k)} + (\omega-\omega_k)\frac{\partial \Delta}{\partial \omega}\bigg|_{\omega_k} + \cancel{i\,\Delta_{IM}} \quad \text{ignore for now}$$

$\Rightarrow$

$$\underbrace{(\omega-\omega_k)}_{i\gamma_k}\frac{\partial \Delta}{\partial \omega}\bigg|_{\omega_k}\hat{\phi}_{k,\omega} = \hat{\phi}^{ext}_{k,\omega}$$

$$\frac{\partial \hat{\phi}^{ext}}{\partial t} = -i\omega\tilde{\phi}^{ext} = -i\omega\left[i\gamma_k \frac{\partial \Delta}{\partial \omega}\bigg|_{\omega_k}\tilde{\phi}_{k,\omega}\right]$$

But, from Poisson eqn.

$$\tilde{\Gamma}_{k,\omega} = \frac{-|k|}{2\pi G}\tilde{\phi}_{k,\omega}\exp\left[i(k\cdot x-\omega t)\right]e^{\gamma_k t}$$

$$\left(\frac{\partial \tilde{\phi}}{\partial t}\right)_{k,\omega} = \gamma_k\,\omega\frac{\partial \Delta}{\partial \omega}\bigg|_{\omega_k}\tilde{\phi}_{k,\omega}\exp\left[i(k\cdot x-\omega t)\right]e^{\gamma_k t}$$

so for wave energy density:

$$\frac{d\mathcal{E}_\omega}{dt} = \frac{\omega}{2}\left(J^* \frac{\partial \tilde{\phi}}{\partial t}\right)$$

$$\mathcal{E}_\omega = \int_{-\infty}^{t} dt' \left(\qquad \right)$$

$$= \int_{-\infty}^{t} dt'\, e^{2\gamma_k t} \left(\frac{-|k|}{4\pi G}\right) \omega_k 2 \frac{\gamma_k}{2} \left.\frac{\partial D}{\partial \omega}\right|_{\omega_k} |\tilde{\phi}_{k,\omega}|^2$$

$$= \int_{-\infty}^{t} dt' (2\gamma_k) e^{2\gamma_k t} \left(\frac{-|k|}{8\pi G}\right) \omega_k \left.\frac{\partial D}{\partial \omega}\right|_{\omega_k} |\tilde{\phi}_{k,\omega}|^2$$

$$= \frac{-|k|}{8\pi G} \omega_k \left.\frac{\partial D}{\partial \omega}\right|_{\omega_k} |\tilde{\phi}_{k,\omega}|^2$$

$$\boxed{\begin{aligned} \mathcal{E}_{w_{\underline{k}}} &= \left(\omega_{\underline{k}} \left.\frac{\partial D}{\partial \omega}\right|_{\omega_{\underline{k}}}\right) \left(\frac{-1}{8\pi G} |f_{g\,k}|^2\right) \\ |f_{g_k}|^2 &= k^2 |\phi_k|^2 \end{aligned}}$$

obvious analogy to

$$\mathcal{E}_{\omega_{\underline{k}}} = \left(\omega_k \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}\right) \frac{|E_k|^2}{8\pi}$$

$$\epsilon = 1 - \omega_p^2/\omega^2$$

$$\omega \frac{\partial \epsilon}{\partial \omega} = 2\frac{\omega_p^2}{\omega^2} = 2$$

→ +1 for field

→ +1 for K.E.

Now $\quad \dfrac{\partial D}{\partial \omega} = \dfrac{-2\pi G \sigma_0 |k| \; 2\bar{\omega}_r}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2} \qquad \tilde{f} = 0\tilde{\phi}$

$$\boxed{\mathcal{E}_w = \frac{\omega_{r_k} \, \bar{\omega}_k}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2} \, \sigma_0 \left(\frac{|\tilde{F}_k|^2}{2} \right)}$$

Spiral wave energy density → obvious $\bar{\omega}_k < 0$, $\mathcal{E}_w < 0$ $(r < r_{co})$

~~can be negative~~ inside co-rotation!

$\bar{\omega}_k > 0$, $\mathcal{E}_w > 0$ $(r > r_{co})$

For wave angular momentum density,

$$\frac{d}{dt} L_\theta = \int_0^{2\pi} d\phi \int_0^\infty r\,dr\, \tilde{\sigma} \left(-\frac{1}{r} \frac{\partial \tilde{\phi}^{ext}}{\partial \phi} \right)$$

⇒ as before : $\left\{ \begin{array}{l} \nabla \phi = \text{ext} \\ \text{and expand} \end{array} \right.$

$$\mathcal{L}_\theta = \frac{-m|k|}{8\pi G} \left(\frac{\partial D}{\partial \omega} \right)\Bigg|_{\omega_k} |\tilde{\phi}_{k,\omega_k}|^2$$

$$= \frac{-m|k|}{8\pi G} \left(\frac{-4\pi G \sigma_0 |k| \bar{\omega}_r}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2} \right) |\tilde{\phi}_{k,\omega_k}|^2$$

$$\Rightarrow \boxed{\mathcal{L}_\theta = \frac{m\bar{\omega}\,\sigma_0}{(\bar{\omega}^2-\kappa^2-k^2c_s^2)^2}\frac{|\tilde{f}_k|^2}{2}} \qquad \left(|\tilde{\rho}_k|^2 = k^2|\tilde{\phi}_k|^2\right)$$

wave angular momentum density

Now

$$\Rightarrow \mathcal{E}_{wave} = \bar{\omega}_{r_k}/m \; \mathcal{L}_{wave}$$

i.e. if $\mathcal{E} = \omega_k N$

$\underline{P} = \underline{k} N \quad \Rightarrow P_\theta = k_\theta N$

$\mathcal{L}_\theta = m N$

$$\mathcal{E}_w / \mathcal{L}_{wave} = (\bar{\omega}_{r_k}/m)$$

$$\Rightarrow \mathcal{E}_{wave} = \frac{\omega_{r_k}\,\bar{\omega}_k\,\sigma_0}{(\bar{\omega}^2-\kappa^2-k^2c_s^2)^2}\frac{|\tilde{f}_k|^2}{2}$$

$$\sim \bar{\omega}_k = (\omega_k - m\Omega(r))\,(\quad)$$

$$\left(\frac{\mathcal{E}_w}{\mathcal{L}_w}\right) > 0 \quad \text{for} \quad \frac{\omega_k}{m} > \Omega(r) \Rightarrow r > r_{co}.$$

$$\left(\frac{\mathcal{E}_w}{\mathcal{L}_w}\right) < 0 \quad \text{for} \quad \frac{\omega_k}{m} < \Omega(r) \Rightarrow r < r_{co}$$

Now, $\mathcal{L}_0 = \dfrac{m\bar{\omega}\,\Sigma_0}{(\bar{\omega}^2-\kappa^2-k^2c_s^2)^2}\,\dfrac{k^2|\tilde{\phi}_k|^2}{2}$

$$v_{gr} = -(\mathrm{sgn}\,k)\left(\pi G\Sigma_0 - |k_r|c_s^2\right)/(\omega - m\Omega)$$

$$v_{gr}\mathcal{L}_0 = -\mathrm{sgn}\,k\,|k_r|\frac{(\pi G\Sigma_0 + |k_r|c_s^2)}{(\bar{\omega}^2-\kappa^2-k^2c_s^2)^2}(m k_r)\Sigma_0\frac{|\tilde{\phi}_k|^2}{2}$$

$\Leftarrow$ wave angular momentum flux

$$= -\left(\pi G\Sigma_0 k_r - k_r^2 c_s^2\right)\frac{(m k_r)\Sigma_0}{(\bar{\omega}^2-\kappa^2-k_r^2c_s^2)^2}\frac{|\tilde{\phi}_k|^2}{2}$$

$$v_{gr}\mathcal{L}_0 = \frac{(m k_r)\Sigma_0|\tilde{\phi}_k|^2}{(\bar{\omega}^2-\kappa^2-k^2c_s^2)^2\,2}\left[+(k_r^2c_s^2 - \pi G\Sigma_0|k_r|)\right]$$

and

$$\Sigma_0 r\langle\tilde{v}_r\tilde{v}_\phi\rangle = \frac{\Sigma_0 m k_r\,|\phi_k|^2}{(\bar{\omega}^2-\kappa^2-k_r^2c_s^2)^2}\left[-2\pi G\Sigma_0|k| + k^2c_s^2\right]|\phi_k|^2$$

modulo diddles, agree!

i.e. propagation of wave thru system (with angular momentum) equivalent to stress.

i.e. waves $\begin{cases} \text{positive energy} & \text{outside co-rotation} \\ \text{negative energy} & \text{inside co-rotation} \end{cases}$

What does this mean?

On wave energy:

- wave energy relative to certain frame ⇒ not Galilean invariant

- Medium active ⇒ supports energy, possibly accessible

i.e. beam-plasma

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{p_b}^2}{(\omega - k v_b)^2}$$

- relative velocity depends on frame ("beam" vs "plasma")
- 'activity' ⟺ beam kinetic energy

⇒ positive energy wave ⟺ excited by adding energy to wave (i.e. $M=0$, cold plasma)

⇒ negative energy wave ⟺ excited by extracting energy from wave (i.e. drive energy more negative, increasing amplitude i.e. $\mathcal{E} \sim \underbrace{\left[\omega \frac{\partial D}{\partial \omega}\right]}_{<0} k^2 |\tilde{\phi}_k|^2$

so
$$\mathcal{E} \downarrow \Leftrightarrow |\tilde{\phi}|^2 \uparrow.$$

Need active medium for negative energy.

Thus, damping of spiral wave encompassing co-rotation, would dissipate:

→ a ⊕ ammount of $\left(\frac{\mathcal{E}}{\mathcal{L}_o}\right)$ for $r > r_{co}$ $(\omega > m\Omega)$

→ a ⊖ ammount of $\left(\frac{\mathcal{E}}{\mathcal{L}_o}\right)$ for $r < r_{co}$ $(\omega < m\Omega)$

→ ⊕ wave $\mathcal{L}_o$ dissipated → positive wave angular momentum transferred to medium → medium angular momentum increases!

→ ⊖ wave $\mathcal{L}_o$ dissipated → negative wave angular momentum transferred to medium → medium angular momentum decreases!

∴ ⊕ $\mathcal{L}_o$, $\mathcal{E}_w$ dissipation bad for accretion

i.e. – increases L_o of disk medium
– accretion requires removal of angular momentum (c.f. L-B rules of two state of single particle at large radius).

⊖ $\mathcal{L}_o$, $\mathcal{E}_w$ dissipation good for accretion
⇒ less angular momentum in disk (as wave $\mathcal{L}_o$ grows at expense of disk)
⇒ L_o dropped ⇒ accretion favored

as accretion : gravitational infall, opposed by angular momentum barrier (ℓ^2/r^2).

⇒

(+) wave L_θ dissipation → ⊕ L_θ in disk matter (less in wave)
→ bad for accretion
→ good for excretion

⊖ wave L_θ dissipation → ⊖ L_θ in disk matter (more in wave)
→ good for accretion
i.e. (reduces centrifugal potential $\sim \ell_\theta^2/r^2$)

i.e.

region	wave	dissipation	effect on L_θ
$r > r_{co}$	$E_w > 0$, $L_{\theta w} > 0$	increases disk matter L_θ	inhibits accretion
$r < r_{co}$	$E_w < 0$, $L_{\theta w} < 0$	reduces disk matter L_θ	enables accretion enhances

i.e. $r > r_{co}$ → spiral drives excretion (inward) L_θ transport
$r < r_{co}$ → spiral drives accretion (outward) L_θ transport

dissipation of spirals tends to inflate disk
$r < r_{co}$ → accrete
$r > r_{co}$ → excrete

(A.) negative energy momentum good for accretion !

(B.) But also, if coupled ⊕, ⊖ energy waves can interact at co-rotation, could:

- initiate positive feedback loop
- amplify spirals

aside → ⊕, ⊖ energy coupling

i.e. $D = D_r + i\, D_I$

$$(\omega - \omega_k)\frac{\partial D_r}{\partial \omega} + i\, D_I = 0$$

$$\gamma_k = -\frac{D_I}{\partial D / \partial \omega} \qquad \left.\frac{\partial D_+}{\partial \omega}\right| > 0 \qquad \left.\frac{\partial D_-}{\partial \omega}\right| < 0$$

for ⊕ with ⊖, $D_{I_+} \sim N_-$ $\qquad N \equiv$ quanta # population

$$\frac{\partial N_+}{\partial t} = \gamma_+ N_+ = -\gamma_0 N_- N_+$$

$$\gamma_+ = -\, d_I^{+} / (\partial D/\partial \omega)_{+} \quad \hookrightarrow > 0$$

$$= -\gamma_0 N_-$$

$$\frac{\partial N_-}{\partial t} = +\gamma_- N_- = \gamma_0 N_+ N_-$$

$$\gamma_- = -\frac{d_I^{-}}{(\partial D/\partial \omega)_-} \quad \hookrightarrow < 0$$

$$= +\gamma_0 N_+$$

$$\frac{\partial}{\partial t}\left[N_+ + N_-\right] = 0 \qquad N_{1,2} = N_+ \pm N_-$$

$$\frac{\partial}{\partial t}\left[N_+ - N_-\right] = -2\gamma_0 N_+ N_-$$

$$= -\gamma_0\left[N_1^2 - N_2^2\right]$$

$$\frac{\partial}{\partial t} N_1 = 0 \qquad N_1 = N_1(0)$$

$$\frac{\partial}{\partial t}\left[N_2\right] = -\gamma_0\left[N_1^2 - N_2^2\right]$$

$$= \gamma_0\left[N_2^2 - N_1^2\right]$$

take $N_1 = 0$ (i.e. $N_{+0} + N_{-0} = 0$)

$$\frac{\partial}{\partial t} N_2 = \gamma_0\left[N_2^2 - N_1(0)^2\right] \simeq \gamma_0 N_2^2$$

$$\frac{dN_2}{N_2^2} = \gamma_0 \qquad -\frac{1}{N_2}\Big|_{N_2(0)}^{N_2(t)} = \gamma_0 t$$

$$-\frac{1}{N_2(t)} + \frac{1}{N_2(0)} = \gamma_0 (t)$$

$$\frac{-N_2(0) + N_2(t)}{N_2(0) N_2(t)} = \gamma_0 t$$

$$-N_2(0) + N_2(t) = \left(N_2(0)\, N_2(t)\right)\gamma t$$

$\therefore \quad N_2(t) = N_2(0) \Big/ \left[1 - N_2(0)\gamma_0 t\right]$

i.e. equal # $\oplus, \ominus$ initially
each grows, explosively

$\Rightarrow$ interaction of $\oplus, \ominus$ energy waves can trigger explosive growth processes.

Aside: Basis of Wave Propagation
(with application to spirals)

observe: $D(k,\omega) = 0 \Rightarrow$ wave // eigenmode
defines

generally, $\tilde{\phi} = \phi_0 e^{i(\theta - \omega t)}$ $\quad \boxed{\Phi = \theta - \omega t \quad \text{phase}}$

so $\underline{k} = \nabla \theta = \underline{\nabla} \Phi$
$\omega = -\partial \Phi / \partial t$

$\therefore$ eikonal equation:

$$D\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial t}, x\right) = D(\underline{k}, \omega, x) = 0$$

$$= D\left(\frac{\partial \Phi}{\partial x}, -\frac{\partial \Phi}{\partial t}, x\right) = 0$$

now, $\left.\dfrac{\partial D}{\partial t}\right|_x = 0$ (taken given)

$$\frac{\partial D}{\partial t} = \frac{\partial D}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial t} + \frac{\partial D}{\partial \omega}\frac{\partial \omega}{\partial t}$$

$$= \frac{\partial D}{\partial \underline{k}} \cdot \frac{\partial}{\partial t}\left(\underline{\nabla}\Phi\right) + \frac{\partial D}{\partial \omega}\frac{\partial^2}{\partial t^2}\left(-\Phi\right)$$

$$= \frac{\partial D}{\partial \underline{k}}\,\underline{\nabla}\left(\frac{\partial \Phi}{\partial t}\right) + \frac{\partial D}{\partial \omega}\,-\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial t}\right)$$

$$\frac{\partial D}{\partial t} = 0 = \frac{\partial D}{\partial \omega}\frac{\partial \omega}{\partial t} - \frac{\partial D}{\partial \underline{k}} \cdot \frac{\partial \omega}{\partial \underline{x}}$$

$$\partial \omega / \partial t - \frac{\partial D / \partial \underline{k}}{\partial D / \partial \omega} \cdot \frac{d\omega}{d\underline{x}} = 0$$

$\Rightarrow$

$$\frac{d\omega}{dt} = 0 \quad \text{on} \quad \boxed{\frac{d\underline{x}}{dt} = -\frac{\partial D / \partial \underline{k}}{\partial D / \partial \omega}}$$

i.e. $d\omega / dt = 0$ specifies ray.

ray has characteristic equation

$$\frac{d\underline{x}}{dt} = -\frac{\partial D / \partial \underline{k}}{\partial D / \partial \omega} \equiv \underline{v}_{gr} \rightarrow \text{specifies ray direction}$$

(skip)

Similarly:

$$\left.\frac{\partial D}{\partial x}\right|_{t=\text{const.}} = 0$$

$$0 = \frac{\partial D}{\partial k} \cdot \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial D}{\partial \omega}\left(\frac{-\partial^2 \Phi}{\partial x \partial t}\right)$$

$$= \frac{\partial D}{\partial k} \cdot \frac{\partial k}{\partial x} - \frac{\partial D}{\partial \omega}\frac{\partial k}{\partial t}$$

$\Rightarrow$

$$-\frac{\partial D/\partial x}{\partial D/\partial \omega} = \frac{\partial k}{\partial t} - \frac{\partial D/\partial k}{\partial D/\partial \omega} \cdot \partial k/\partial x = \frac{dk}{dt} = 0$$

$$\text{on } \frac{dx}{dt} = \frac{-\partial D/\partial k}{\partial D/\partial \omega} = \frac{d\omega}{dk}.$$

$\therefore$ $\left. dk/dt \right|_t = 0$ both on $\dfrac{dx}{dt} = V_{gr} = \dfrac{d\omega}{dk}$

$\left. d\omega/dt \right|_x = 0$

For spirals,

$$D(k,\omega) = 1 + \frac{2\pi G\sigma_0 |k|}{\bar{\omega}^2 - \kappa^2 - k^2 c_s^2}$$

$$-\partial D/\partial k = -\frac{2\pi G\sigma_0\, \mathrm{sgn}\, k}{\bar{\omega}^2 - \kappa^2 - k^2 c_s^2} - \frac{2\pi G\sigma_0 |k|\,(+2k c_s^2)}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}$$

$$\partial D/\partial \omega = \frac{-2\bar{\omega}\,(2\pi G\sigma_0 |k|)}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}$$

$$\frac{-\partial D/\partial k}{\partial D/\partial \omega} = \frac{\dfrac{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)(-\cancel{2\pi G\sigma_0}\,\mathrm{sgn}\,k) - \cancel{2\pi G\sigma_0}|k|\,(2k c_s^2)}{\cancel{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}}}{\dfrac{-2\bar{\omega}\,(\cancel{2\pi G\sigma_0}|k|)}{\cancel{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}}}$$

$$\bar{\omega}^2 = \kappa^2 + k^2 c_s^2 - 2\pi G\sigma_0 |k|$$

$$\frac{-\partial D/\partial k}{\partial D/\partial \omega} = \frac{+\cancel{2}\pi G\sigma_0 \cancel{|k|}\,(\mathrm{sgn}\,k) - \cancel{2}k\cancel{|k|}c_s^2}{-\cancel{2}\bar{\omega}\,\cancel{|k|}}$$

$$= -\mathrm{sgn}\,k \left[\frac{\pi G\sigma_0 - |k_r| c_s^2}{\bar{\omega}}\right]$$

$$\boxed{v_{gr} = \mathbf{v}_g \cdot \hat{r} = -(\mathrm{sgn}\, k)\left(\pi G \Sigma_0 - |k_r| c_s^2\right) / (\omega - m\Omega)}$$

where:

$\Rightarrow v_{gr} \sim \left(-\mathrm{sgn}\, k\right)\left(\pi G \Sigma_0 - |k_r| c_s^2\right)(\omega - m\Omega)^{-1}$

$$\begin{pmatrix} \ominus \text{ trailing} \\ \oplus \text{ leading} \end{pmatrix} \begin{pmatrix} \oplus \text{ long wave} \\ \ominus \text{ short wave} \end{pmatrix} \begin{pmatrix} r > r_{co} \ \oplus \\ r < r_{co} \ \ominus \end{pmatrix}$$

$$k \lessgtr \left(\pi G \Sigma_0 / c_s^2\right)$$

thus

long trailing spiral:

$\Rightarrow$ propagates inward for $r > r_{co}$

$\Rightarrow$ propagates outward for $r < r_{co}$

$\therefore$ long trailing spiral always propagates toward co-rotation.

short trailing spiral:

$\Rightarrow$ propagates outward for $r > r_{co}$

propagates inward for $r < r_{co}$

$\therefore$ short trailing spiral always propagates away from co-rotation.